

# The conformal Killing spinor initial data equations

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## Abstract

We obtain necessary and sufficient conditions for an initial data set for the conformal Einstein field equations to give rise to a spacetime development in possession of a Killing spinor. This constitutes the conformal analogue of the Killing spinor initial data equations derived in [16]. The fact that the conformal Einstein field equations are used in our derivation allows for the possibility that the initial hypersurface be (part of) the conformal boundary  $\mathcal{I}$ . For conciseness, these conditions are derived assuming that the initial hypersurface is spacelike. Consequently, these equations encode necessary and sufficient conditions for the existence of a Killing spinor in the development of asymptotic initial data on spacelike components of  $\mathcal{I}$ .

## 1 Introduction

The discussion of symmetries in General Relativity is ubiquitous. From the question of integrability of the geodesic equations to the existence of explicit solutions to the Einstein field equations and the black hole uniqueness problem, symmetries always play an important role. Symmetry assumptions are usually incorporated into the Einstein field equations —which in vacuum read

$$\tilde{R}_{ab} = \lambda \tilde{g}_{ab}, \quad (1)$$

through the use of Killing vectors. From the spacetime point of view, the existence of Killing vectors allows one to perform *symmetry reductions* of the Einstein field equations —see for instance [33]. This approach has been exploited in classical uniqueness results such as [27]. Closely related to the black hole uniqueness problem, characterisations and classifications of solutions to the Einstein field equations usually exploit the symmetries of the spacetime in one way or another, e.g., in the characterisations of the Kerr spacetime via the *Mars-Simon tensor* —see [17, 18, 28]. On the other hand, from the point of view of the Cauchy problem, symmetry assumptions should be imposed only at the level of initial data. In this regard, symmetry assumptions can be phrased in terms of the *Killing vector initial data*. The Killing vector initial data equations constitute a set of conditions that an initial data set  $(\tilde{S}, \tilde{h}, \tilde{K})$  for the Einstein field equations has to satisfy to ensure that the development will contain a Killing vector —see [6]. Nevertheless, despite the fact that the existence of Killing vector plays a central role in the discussion of the symmetries, the existence of Killing vectors is sometimes not enough to encode all the symmetries and conserved quantities that a spacetime can possess, e.g., the Carter constant in the Kerr spacetime. To unravel some of these *hidden symmetries* one has to analyse the existence of a more fundamental type of objects; *Killing spinors*  $\tilde{\kappa}_{AB}$  —in vacuum spacetimes, the existence of a Killing spinor directly implies the existence of a Killing vector. The *Killing spinor initial data equations* have been

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derived in the *physical framework* —governed by the Einstein field equations— in [16]. These equations have been successfully employed in the construction of a geometric invariant which detects whether or not an initial data set corresponds to initial data for the Kerr spacetime —see [2, 3, 4]. This analysis has also been extended to include suitable classes of matter —see [7] for an analogous characterisation of initial data for the Kerr-Newman spacetime. In these characterisations, some asymptotic conditions on the initial data are required. These conditions usually take the form of decay assumptions on  $\tilde{\mathbf{h}}$ ,  $\tilde{\mathbf{K}}$  and  $\tilde{\mathbf{\kappa}}$  on  $\tilde{\mathcal{S}}$ , given in terms of asymptotically Cartesian coordinates. Nonetheless, in other approaches, the asymptotic behaviour of the spacetime can be studied in a geometric way through conformal compactifications. The latter is sometimes referred as the Penrose proposal. In this approach one starts with a *physical spacetime*  $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$  where  $\tilde{\mathcal{M}}$  is a 4-dimensional manifold and  $\tilde{\mathbf{g}}$  is a Lorentzian metric which is a solution to the Einstein field equations. Then, one introduces a *unphysical spacetime*  $(\mathcal{M}, \mathbf{g})$  into which  $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$  is conformally embedded. Accordingly, there exists an embedding  $\varphi : \tilde{\mathcal{M}} \rightarrow \mathcal{M}$  such that

$$\varphi^* \mathbf{g} = \Xi^2 \tilde{\mathbf{g}}. \quad (2)$$

By suitably choosing the *conformal factor*  $\Xi$  the metric  $\mathbf{g}$  may be well defined at the points where  $\Xi = 0$ . In such cases, the set of points for where the conformal factor vanishes is at infinity from the physical spacetime perspective. The set

$$\mathcal{I} \equiv \{p \in \mathcal{M} \mid \Xi(p) = 0, \mathbf{d}\Xi(p) \neq 0\}$$

is called the conformal boundary. However, it can be readily verified that the Einstein field equations are not conformally invariant. Moreover, a direct computation using the conformal transformation formula for the Ricci tensor shows that the vacuum Einstein field equations (1), lead to an equation which is formally singular at the conformal boundary. An approach to deal with this problem was given in [9] where a regular set of equations for the unphysical metric was derived. These equations are known as the *conformal Einstein field equations*. The crucial property of these equations is that they are regular at the points where  $\Xi = 0$  and a solution thereof implies whenever  $\Xi \neq 0$  a solution to the Einstein field equations —see [9, 11] and [32] for an comprehensive discussion. There are three ways in which these equations can be presented, the metric, the frame and spinorial formulations. These equations have been mainly used in the stability analysis of spacetimes —see for instance [13, 12] for the proof of the global and semiglobal non-linear stability of the de Sitter and Minkowski spacetimes, respectively.

A conformal version of the Killing vector initial data equations using the metric formulation of the conformal Einstein field equations has been obtained in [24]. In the latter reference, intrinsic conditions on an initial hypersurface  $\mathcal{S} \subset \mathcal{M}$  of the unphysical spacetime are found such that the development of the data —in the unphysical setting the evolution is governed by the conformal Einstein field equations— gives rise to a conformal Killing vector of the unphysical spacetime  $(\mathcal{M}, \mathbf{g})$  which, in turn, corresponds to a Killing vector of the physical spacetime  $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$ . Notice that this approach, in particular, allows  $\mathcal{S}$  to be determined by  $\Xi = 0$  so that it corresponds to the conformal boundary  $\mathcal{I}$ . The unphysical Killing vector initial data equations have been derived for the characteristic initial value problem on a cone in [24] and on a spacelike conformal boundary in [23].

For applications involving the the conformal Einstein field equations —say in its spinorial formulation, one frequently has to fix the gauge and write the equations in components. Despite the fact that, at first glance, the conformal Einstein field equations expressed in components with respect to an arbitrary spin frame seem to be overwhelmingly complicated, as shown in [15], symmetry assumptions (spherical symmetry in the latter case) greatly reduce the number of equations to be analysed. In the case of Petrov type D spacetimes, e.g. the Kerr-de Sitter spacetime, the symmetries of the spacetime are closely related to the existence of Killing spinors. Therefore, a natural question in this setting is whether a conformal version of the Killing spinor initial data equations introduced in [16] can be found. In other words, what are the extra conditions that one has to impose on an initial data set for the conformal Einstein field equations so that the arising development contains a Killing spinor? This question is answered in this article by deriving such conditions which we call the *conformal Killing spinor initial data equations*

Despite the fact that the Killing spinor equation is conformally invariant, it is not a priori clear whether the conditions of [16, 3] may be translated directly into the unphysical setting. Indeed, one expects this not to be the case, since the Einstein field equations are not conformally invariant. Moreover, one consideration that is exploited in the discussion of [16] is based on the fact that, on an Einstein spacetime  $(\tilde{\mathcal{M}}, \tilde{g})$ , a Killing spinor  $\tilde{\kappa}_{AB}$  gives rise to a Killing vector  $\tilde{\xi}_a$  whose spinorial counterpart is given by  $\tilde{\xi}_{AA'} = \tilde{\nabla}_{A'}^Q \tilde{\kappa}_{QA}$ . Nevertheless, this property does not hold in general. In other words, if  $(\mathcal{M}, g)$ , where  $g$  is not assumed to satisfy the Einstein field equations, possess a Killing spinor  $\kappa_{AB}$ , then the analogous concomitant  $\xi_{AA'} = \nabla_{A'}^Q \kappa_{QA}$  does not correspond to a Killing vector—not even a conformal Killing vector. This situation is not ameliorated if one assumes that  $(\mathcal{M}, g)$  satisfies the conformal Einstein field equations. Nevertheless, as discussed in this article, in the latter case one can show that using the conformal factor  $\Xi$ , the Killing spinor  $\kappa_{AB}$  and the *auxiliary vector*  $\xi_a$ , one can construct a conformal Killing vector  $X_a$  associated to a Killing vector  $\tilde{X}_a$  of the physical spacetime  $(\tilde{\mathcal{M}}, \tilde{g})$ . The conditions of [16, 3] may be recovered from the results presented here by setting  $\Xi = 1$ .

An interesting feature of our analysis is the fact that we make use of an alternative representation of the conformal Einstein field equations. In principle, one could use the standard representation of the conformal Einstein field equations, however, some experimentation reveals that the latter approach leads to Fuchsian systems of equations—formally singular at the conformal boundary—for quantities associated to the Killing spinor. For conciseness, the conformal Killing initial data equations are obtained on a spacelike hypersurface  $\mathcal{S}$ . Nonetheless, a similar computation can be performed on an hypersurface  $\mathcal{S}$  with a different causal character. The conditions found in this article have potential applications for the black hole uniqueness problem. In particular, they can be used for an asymptotic characterisation of the Kerr-de Sitter spacetime analogous to [19] in terms of the existence of Killing spinors at the conformal boundary  $\mathcal{I}$ .

The main results of this article are summarised informally in the following:

**Theorem.** *If the conformal Killing spinor initial data equations (C1)-(C3) are satisfied on an open set  $\mathcal{U} \subset \mathcal{S}$ , where  $\mathcal{S}$  is a spacelike hypersurface on which initial data for the (alternative) conformal Einstein field equations has been prescribed, then, the domain of dependence of  $\mathcal{U}$  possesses a Killing spinor. Moreover, assuming conditions (C1), (C2) to hold, condition (C3) is equivalent to the vanishing of certain components of the Cotton spinor, with respect to a suitably-adapted spin dyad.*

A precise formulation is the content of Theorem 2 and Proposition 2.

Involved computations throughout this article were facilitated through the suite **xAct** in **Mathematica**. Note that since the existence of a spinor structure is guaranteed for globally-hyperbolic spacetimes—see Proposition 4 in [32]—the use of spinors is not overly restrictive.

## Overview of the article

Section 2 gives an overview of Killing spinors along with their conformal properties. In Section 3 we describe the conformal Einstein field equations in two different representations, for later use; Section 4 introduces the main objects of interest in the propagation of Killing spinor data, namely the *Killing spinor zero-quantities*. In Section 5 we construct conformally-regular wave equations for the zero-quantities, leading to necessary and sufficient conditions for the existence of a Killing spinor—see Proposition 1. In Section 6 the latter conditions and the space spinor formalism are used to obtain the conformal Killing spinor initial data equations on spacelike hypersurfaces—see Theorem 2. Finally, in Section 7 the latter equations are analysed with respect to an adapted spin dyad and the implied restrictions on the Cotton spinor are presented—see Proposition 2.

## Notation and conventions

Upper case Latin indices  $ABC\dots A'B'C'$  will be used as abstract indices of the *spacetime spinor* algebra, and the bold numerals  $\mathbf{012}\dots$  denote components with respect to a fixed spin dyad  $o^A \equiv \epsilon_0^A, \iota^A \equiv \epsilon_1^A$ —see Penrose & Rindler [25] for further details. Although spinor notation

will be preferred, for certain computations tensors will be employed. Lower case Latin indices  $a, b, c, \dots$  will be used as abstract tensor indices. For tensors, our curvature conventions are fixed by

$$\nabla_a \nabla_b \kappa^c - \nabla_b \nabla_a \kappa^c = R_{ab}{}^c{}_d \kappa^d.$$

For spinors, the curvature conventions are fixed via the spinorial Ricci identities which will be written in accordance with the above convention for tensors. To see this, recall that the commutator of covariant derivatives  $[\nabla_{AA'}, \nabla_{BB'}]$  can be expressed in terms of the symmetric operator  $\square_{AB}$  as

$$[\nabla_{AA'}, \nabla_{BB'}] = \epsilon_{AB} \square_{A'B'} + \epsilon_{A'B'} \square_{AB}$$

where

$$\square_{AB} \equiv \nabla_{Q'(A} \nabla_{B)}{}^{Q'}.$$

The action of the symmetric operator  $\square_{AB}$  on valence-1 spinors is encoded in the spinorial Ricci identities

$$\square_{AB} \xi_C = -\Psi_{ABCD} \xi^D + 2\Lambda \xi_{(A} \epsilon_{B)C}, \quad (3a)$$

$$\square_{A'B'} \xi_C = -\xi^A \Phi_{CAA'B'}, \quad (3b)$$

where  $\Psi_{ABCD}$  and  $\Phi_{AA'BB'}$  and  $\Lambda$  are curvature spinors. The above identities can be extended to higher valence spinors in an analogous way —see [30] for further discussion on these identities using different conventions to the ones used in this article. A related identity which will be systematically used in the following discussion is

$$\nabla_{AQ'} \nabla_B{}^{Q'} = \square_{AB} + \frac{1}{2} \epsilon_{AB} \square, \quad (4)$$

where  $\square_{AB}$  is the symmetric operator defined above and  $\square \equiv \nabla_{AA'} \nabla^{AA'}$ .

## 2 Killing spinors

To start the discussion it is convenient to introduce some notation and definitions. Let  $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$  be a 4-dimensional manifold equipped with a Lorentzian metric  $\tilde{\mathbf{g}}$  and denote by  $\tilde{\nabla}$  its associated Levi-Civita connection. For the time being  $\tilde{\mathbf{g}}$  is not assumed to be a solution to the Einstein field equations (1). A symmetric valence-2 spinor,  $\tilde{\kappa}_{AB} = \tilde{\kappa}_{(AB)}$ , is a *Killing spinor* if it satisfies the equation

$$\tilde{\nabla}_{A'(A} \tilde{\kappa}_{BC)} = 0. \quad (5)$$

The Killing spinor equation is, in general, overdetermined; in particular, it implies the so-called *Buchdahl constraint*:

$$\tilde{\kappa}^Q{}_{(A} \Psi_{BCD)Q} = 0,$$

where  $\Psi_{ABCD}$  denotes the conformally invariant Weyl spinor. The latter condition restricts  $\Psi_{ABCD}$  to be algebraically special —Petrov type D, N or O. Another relevant property of the Killing spinor equation is that it is conformally-invariant, in other words if  $\mathbf{g}$  is conformally related to  $\tilde{\mathbf{g}}$ , namely  $\mathbf{g} = \Xi^2 \tilde{\mathbf{g}}$  then  $\kappa_{AB} = \Xi^2 \tilde{\kappa}_{AB}$  satisfies

$$\nabla_{A'(A} \kappa_{BC)} = 0,$$

where  $\nabla$  is the Levi-Civita connection of  $\mathbf{g}$ . For general  $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$  the existence of a Killing spinor is, however, not related directly to the existence of a Killing vector. Nevertheless, if one assumes that  $\tilde{\mathbf{g}}$  satisfies the vacuum Einstein field equations (1) then the concomitant

$$\tilde{\xi}_{AA'} \equiv \tilde{\nabla}^B{}_{A'} \tilde{\kappa}_{AB},$$

represents the spinorial counterpart of a complex Killing vector of the spacetime  $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$  —see [16] for further discussion. Moreover, if  $\tilde{\xi}_{AA'}$  is Hermitian, i.e.,  $\tilde{\bar{\xi}}_{AA'} = \tilde{\xi}_{AA'}$ , then one can construct the

spinorial counterpart of a *Killing–Yano tensor*  $\tilde{\Upsilon}_{ab}$  —i.e. an antisymmetric 2–tensor satisfying  $\tilde{\nabla}_{(a}\tilde{\Upsilon}_{b)c} = 0$ — as follows

$$\tilde{\Upsilon}_{AA'BB'} = i(\tilde{\kappa}_{AB}\tilde{\epsilon}_{A'B'} - \tilde{\kappa}_{A'B'}\tilde{\epsilon}_{AB}).$$

Conversely, given a Killing–Yano tensor, one can construct a Killing spinor —see [7, 20, 26].

In the sequel  $(\tilde{\mathcal{M}}, \tilde{g})$  will be reserved to denote the *physical spacetime*, in other words, the symbol  $\tilde{\phantom{x}}$  will be added to those fields associated with a solution  $\tilde{g}$  to the vacuum Einstein field equations (1). Similarly  $(\mathcal{M}, g)$  will be used to represent the *unphysical spacetime* related to  $(\tilde{\mathcal{M}}, \tilde{g})$  via  $g = \Xi^2 \tilde{g}$ . —in a slight abuse of notation  $\varphi(\tilde{\mathcal{M}})$  and  $\mathcal{M}$  will be identified so that the mapping  $\varphi : \tilde{\mathcal{M}} \rightarrow \mathcal{M}$  can be omitted.

### 3 The conformal Einstein field equations

As discussed in the introduction, for the derivation of the conformal Killing initial data equations, an appropriate formulation of the conformal Einstein field equations will be required. In this section we begin, for the sake of completeness, with a discussion of the standard conformal field equations (CFEs) originally introduced in [9] by H. Friedrich —see also [32]. Then, an alternative formulation to these equations are presented. The main benefit of these equations, which we refer to as the *alternative CFEs*, in our context is that the so-called *rescaled Weyl tensor* is replaced by the *Weyl tensor* through the introduction of the *Cotton tensor* as an additional unknown. This latter approach was first proposed in [24] by T. Paetz.

The use of the alternative CFEs is vindicated by our final result which indicates that the existence of a Killing spinor necessarily places restrictions on the components of the Cotton spinor at the level of the initial data —see Theorem 1, Proposition 2.

#### 3.1 The standard conformal Einstein field equations

The conformal Einstein field equations are a conformal formulation of the Einstein field equations. In other words, given a spacetime  $(\tilde{\mathcal{M}}, \tilde{g})$  satisfying the Einstein field equations, the conformal Einstein field equations encode a system of differential conditions for the curvature and concomitants of the conformal factor associated with  $(\mathcal{M}, g)$  where  $g = \Xi^2 \tilde{g}$ . The key property of these equations is that they are regular even at the conformal boundary  $\mathcal{I}$ , where  $\Xi = 0$ . This formulation of the conformal Einstein field equations was first given in [9] —see also [32] for a comprehensive discussion.

The metric version of the standard vacuum conformal Einstein field equations are encoded in the following zero-quantities —see [9, 8, 10, 11]:

$$Z_{ab} \equiv \nabla_a \nabla_b \Xi + \Xi L_{ab} - s g_{ab} = 0, \quad (6a)$$

$$Z_a \equiv \nabla_a s + L_{ac} \nabla^c \Xi = 0, \quad (6b)$$

$$\delta_{bac} \equiv \nabla_b L_{ac} - \nabla_a L_{bc} - d_{abcd} \nabla^d \Xi = 0, \quad (6c)$$

$$\lambda_{abc} \equiv \nabla_e d_{abc}{}^e = 0, \quad (6d)$$

$$Z \equiv \lambda - 6\Xi s + 3\nabla_a \Xi \nabla^a \Xi \quad (6e)$$

where  $\Xi$  is the conformal factor,  $L_{ab}$  is the Schouten tensor, defined in terms of the Ricci tensor  $R_{ab}$  and the Ricci scalar  $R$  via

$$L_{ab} = \frac{1}{2}R_{ab} - \frac{1}{12}Rg_{ab}, \quad (7)$$

$s$  is the so-called *Friedrich scalar* defined as

$$s \equiv \frac{1}{4}\nabla_a \nabla^a \Xi + \frac{1}{24}R\Xi \quad (8)$$

and  $d^a{}_{bcd}$  denotes the *rescaled Weyl tensor*, defined as

$$d^a{}_{bcd} = \Xi^{-1}C^a{}_{bcd},$$

where  $C^a{}_{bcd}$  denotes the Weyl tensor. The geometric meaning of these zero-quantities is the following: The equation  $Z_{ab} = 0$  encodes the conformal transformation law between  $R_{ab}$  and  $\tilde{R}_{ab}$ . The equation  $Z_a = 0$  is obtained considering  $\nabla^a Z_{ab}$  and commuting covariant derivatives. Equations  $\delta_{abc} = 0$  and  $\lambda_{abc} = 0$  encode the contracted second Bianchi identity. Finally,  $Z = 0$  is a constraint in the sense that if it is verified at one point  $p \in \mathcal{M}$  then  $Z = 0$  holds in  $\mathcal{M}$  by virtue of the previous equations. A solution to the metric conformal Einstein field equations consist of a collection of fields

$$\{g_{ab}, \Xi, \nabla_a \Xi, s, L_{ab}, d_{abcd}\}$$

satisfying

$$Z_{ab} = 0, \quad Z_a = 0, \quad \delta_{abc} = 0, \quad \lambda_{abc} = 0, \quad Z = 0.$$

**Remark 1.** In the metric formulation of the standard conformal Einstein field equations one needs to supplement the system encoded in the zero quantities defined above with an equation for the unphysical metric  $g_{ab}$ . To do so, one considers equation (7) expressed in some local coordinates  $(x^\mu)$ . Recalling that in local coordinates the components of the Ricci tensor can be written as second order derivatives of the metric, one obtains the required equation for the unphysical metric. This observation applies also for the subsequent discussion of the alternative conformal Einstein field equations.

### 3.2 The alternative conformal Einstein field equations

In the current formulations of the conformal Einstein field equations the rescaled Weyl tensor  $d^a{}_{bcd}$  plays a central role in the discussion. Nevertheless in [22] an alternative approach was outlined, whereby the central object of interest is the Weyl tensor itself. In doing so, one must also introduce the Cotton tensor as an unknown. In [22] a set of wave equations for these unknowns is constructed. Here we follow a similar approach, but rather than deriving second-order equations for the conformal fields, we will obtain a closed system of equations which are (apart from the equation for the conformal factor, (14a)) of first-order. We will call the resulting equations, along with their spinorial equivalent, the *alternative CFEs*.

Considering  $\nabla^a \delta_{abc} = 0$ , with  $\delta_{abc}$  as given in expression (6c), one obtains the following wave equation for the Schouten tensor

$$\square L_{bc} = 4L_b{}^a L_{ca} - L_{ad} L^{ad} g_{bc} - 2L^{ad} C_{bacd} + \frac{1}{6} \nabla_c \nabla_b R. \quad (9)$$

Recalling the definition of the Cotton tensor in terms of the Schouten tensor

$$Y_{abc} = 2(-\nabla_a L_{bc} + \nabla_b L_{ac}) \quad (10)$$

and using equations (9) and (10) a computation shows that

$$\nabla_a Y_b{}^a{}_c = -2L^{ad} C_{bacd}. \quad (11)$$

Therefore, to close the system one needs to find an equation for the Weyl tensor. To do so, one can use the second Bianchi identity

$$\nabla_{[a} R_{bf]c}{}^d = 0$$

and the decomposition of the Riemann tensor in terms of the Weyl and Schouten tensors

$$R_{abc}{}^d = \delta_b{}^d L_{ac} - \delta_a{}^d L_{bc} + L_b{}^d g_{ac} - L_a{}^d g_{bc} + C_{abc}{}^d,$$

to obtain

$$\nabla_{[a} C_{bf]c}{}^d - 2g_{[a|c|} \nabla_b L_{f]}{}^d + 2g_{[a}{}^d \nabla_b L_{f]c} = 0. \quad (12)$$

Using equations (12) and (10) one can rewrite equation (12) as

$$\nabla_{[a} C_{bf]c}{}^d = \frac{1}{2} Y_{[ab|c|} g_{f]}{}^d - \frac{1}{2} Y_{[ab}{}^d g_{f]c}. \quad (13)$$

Consequently, one can replace the zero-quantities associated with the rescaled Weyl tensor using equations (13), (10) and (11), obtaining then an alternative version of the conformal Einstein field equations. The equations so obtained are encoded in the vanishing of the following zero-quantities.

$$Z_{ab} \equiv \nabla_a \nabla_b \Xi + \Xi L_{ab} - s g_{ab} = 0, \quad (14a)$$

$$Z_a \equiv \nabla_a s + L_{ac} \nabla^c \Xi = 0, \quad (14b)$$

$$Z \equiv \lambda - 6\Xi s + 3\nabla_a \Xi \nabla^a \Xi, \quad (14c)$$

$$\Delta_{bac} \equiv \nabla_b L_{ac} - \nabla_a L_{bc} - \frac{1}{2} Y_{abc}, \quad (14d)$$

$$\Pi_{bc} \equiv \nabla_a Y_b{}^a{}_c + 2L^{ad} C_{bacd}, \quad (14e)$$

$$\Lambda_{abcd}{}^e = 2\nabla_{[a} C_{bc]d}{}^e - Y_{[ab|d|} g_{c]}{}^e + Y_{[ab}{}^e g_{c]d} \quad (14f)$$

Notice that in this alternative representation of the conformal Einstein field equations the rescaled Weyl tensor does not appear. Instead, the Weyl tensor  $C_{abcd}$  and the Cotton tensor  $Y_{abc}$  are now part of the unknowns. Observe that the definition of the Cotton tensor in terms of derivatives of the Schouten tensor is encoded in equation (14d). A solution to the alternative conformal Einstein field equations consists of a collection of fields

$$\{g_{ab}, \Xi, \nabla_a \Xi, s, L_{ab}, C_{abcd}, Y_{abc}\} \quad (15)$$

satisfying

$$Z_{ab} = 0, \quad Z_a = 0, \quad \Delta_{abc} = 0, \quad \Pi_{bc} = 0, \quad \Lambda_{abcde} = 0.$$

**Remark 2.** Note that, by construction, any solution to the (alternative) CFEs with  $\Xi = 1$  corresponds to a solution of the Einstein field equations. Conversely, given a solution to the Einstein field equations, there corresponds a family of conformally-related solutions to the (alternative) CFEs.

In view of the subsequent analysis of the Killing spinor equation it is convenient to formulate the above system in spinorial form. Similar to the case of the standard conformal Einstein field equations, the spinorial formulation allows one to identify in a clearer way the structure of the equations. To obtain the spinorial formulation of the the zero-quantities (14a)-(14f) recall that the spinorial counterpart of the Weyl tensor can be decomposed as

$$C_{AA'BB'CC'DD'} = \bar{\Psi}_{A'B'C'D'} \epsilon_{AB} \epsilon_{CD} + \Psi_{ABCD} \bar{\epsilon}_{A'B'} \bar{\epsilon}_{C'D'},$$

where  $\Psi_{ABCD}$  is the Weyl spinor. Similarly, the irreducible decomposition of the Cotton spinor  $Y_{AA'BB'CC'}$  given by

$$Y_{AA'BB'CC'} = \bar{Y}_{A'B'C'C} \epsilon_{AB} + Y_{ABCC'} \bar{\epsilon}_{A'B'}, \quad (16)$$

where

$$Y_{ABCC'} = \frac{1}{2} Y_{(A|Q'|B}{}^{Q'}{}_{C)C'}. \quad (17)$$

Additionally, the Schouten spinor  $L_{AA'BB'}$  can be expressed in terms of the tracefree Ricci spinor  $\Phi_{AA'BB'}$  and  $\Lambda = \frac{1}{24} R$ ;

$$L_{AA'BB'} = -\Phi_{ABA'B'} + \Lambda \epsilon_{AB} \epsilon_{A'B'}. \quad (18)$$

With these decompositions at hand, the spinorial formulation of the above equations can be expressed as

$$Z_{AA'BB'} = 0, \quad Z_{AA'} = 0, \quad \Delta_{ABCC'} = 0, \quad \Pi_{BB'CC'}, \quad \Lambda_{C'BCF} = 0, \quad (19)$$

where

$$Z_{AA'BB'} \equiv \nabla_{BB'} \nabla_{AA'} \Xi - \Xi \Phi_{ABA'B'} \epsilon_{A'B'} + \Xi \Lambda \epsilon_{AB} \epsilon_{A'B'} - s \epsilon_{AB} \epsilon_{A'B'}, \quad (20a)$$

$$Z_{AA'} \equiv \nabla_{AA'} s - \Phi_{AA'}{}^{BB'} \nabla_{BB'} \Xi + \Lambda \nabla_{AA'} \Xi, \quad (20b)$$

$$\Delta_{ABCC'} \equiv -Y_{ABCC'} + \nabla_{AA'} \Phi_{BCC'}{}^{A'} + \epsilon_{BC} \nabla_{AC'} \Lambda + \nabla_{BA'} \Phi_{ACC'}{}^{A'} + \epsilon_{AC} \nabla_{BC'} \Lambda, \quad (20c)$$

$$\Pi_{AA'BB'} \equiv -2\Phi^{CD}{}_{A'B'} \Psi_{ABCD} + 2\Phi_{AB}{}^{C'D'} \bar{\Psi}_{A'B'C'D'} + \nabla_{CC'} Y_{AA'}{}^{CC'}{}_{BB'}, \quad (20d)$$

$$\Lambda_{C'BCF} \equiv -\frac{1}{2}Y_{BCFC'} + \nabla_{AC'}\Psi_{BCF}{}^A. \quad (20e)$$

Notice that the zero-quantity  $\Pi_{AA'BB'}$  can alternatively be written in terms of the reduced Cotton spinor  $Y_{ABCC'}$  as

$$\Pi_{AA'BB'} = -2\Phi^{CD}{}_{A'B'}\Psi_{ABCD} - 2\Phi_{AB}{}^{C'D'}\bar{\Psi}_{A'B'C'D'} + \nabla_{AC'}\bar{Y}_{A'}{}^{C'}{}_{B'B} + \nabla_{CA'}Y_A{}^C{}_{BB'}.$$

Furthermore, observe that a trace of the latter equation implies

$$\Pi^A{}_{A'AB'} = -\nabla_{AC'}\bar{Y}_{A'B'}{}^{C'A}.$$

Its worth noticing that in the formulation of the conformal Einstein field equations, the Ricci scalar  $\Lambda$  is not part of the unknowns as it represents, the so-called, the *conformal gauge source function* —see [11, 14, 32] for further discussion. The geometric meaning of the zero-quantities (20a)-(20e) is analogous to their tensorial counterparts. In particular, the Bianchi identities may be recovered from the alternative conformal Einstein field equations by taking suitable contractions of the zero-quantities  $\Delta_{ABCC'}$  and  $\Lambda_{ABB'C}$ :

$$\nabla_{AA'}\Phi_{BC'}{}^{AA'} + 3\nabla_{BC'}\Lambda = -\Delta_B{}^A{}_{AC'} \quad (21a)$$

$$\nabla_{AC'}\Psi_{BCF}{}^A + \nabla_{(B}{}^{Q'}\Phi_{CF)C'Q'} = -\frac{1}{2}\Delta_{(BCF)C'} + \Lambda_{C'(BCF)} \quad (21b)$$

By *initial data for the alternative CFEs* we mean the restriction to an hypersurface  $\mathcal{S} \subset \mathcal{M}$  of a collection of fields (15), satisfying the constraint equations implied by (19). It will not be necessary for our purposes to study the constraints equations in detail; indeed, the only constraint that will be of interest —see Section 6— is the following

$$\mathcal{D}^{PQ}\Psi_{ABPQ} - \frac{1}{2}Y_{AB}{}^Q{}_Q = 0$$

which follows from  $\Lambda_{A'ABC} = 0$ . Since the (alternative) conformal Einstein field equations imply a solution to the Einstein field equations (1) whenever  $\Xi \neq 0$ , we will refer to the development such an initial data set simply as a *spacetime development*.

## 4 Killing spinor zero-quantities

For the subsequent discussion it is convenient to introduce the following spinors

$$H_{A'ABC} \equiv 3\nabla_{A'(A}\kappa_{BC)}, \quad (22a)$$

$$S_{AA'BB'} \equiv \nabla_{QA'}H_{B'}{}^Q{}_{AB}, \quad (22b)$$

$$B_{ABCD} \equiv -\frac{1}{6}\nabla_{Q'(A}H^{Q'}{}_{BCD)}. \quad (22c)$$

Observe that if  $(\mathcal{M}, g)$  admits a Killing spinor  $\kappa_{AB}$  then, by definition, one has

$$H_{A'ABC} = 0, \quad S_{AA'BB'} = 0, \quad B_{ABCD} = 0.$$

To see the geometric significance of the above defined zero-quantities it is convenient to introduce the Hermitian spinor

$$\xi_{AA'} \equiv \nabla^B{}_{A'}\kappa_{AB}. \quad (23)$$

Observe that the zero-quantity  $S_{AA'BB'}$  can be written in terms of  $\xi_{AA'}$  as

$$S_{CC'DD'} = -6\kappa_{(D}{}^A\Phi_{C)AC'D'} - \nabla_{CC'}\xi_{DD'} - \nabla_{DD'}\xi_{CC'}. \quad (24)$$

Notice that, if  $\Phi_{ABA'B'}$  vanishes then  $S_{AA'BB'}$  reduces to the Killing vector equation and  $\xi_{AA'}$  corresponds to the spinorial counterpart of a Killing vector. To clarify this point further observe that, as a consequence of the conformal properties of the Killing spinor equation, if  $\kappa_{AB}$  is a Killing spinor in the unphysical spacetime then

$$\tilde{\kappa}_{AB} = \frac{1}{\Xi^2}\kappa_{AB}$$



is a Killing spinor of the physical spacetime whenever  $\Xi$  is not vanishing. As discussed before, the physical Killing spinor  $\tilde{\kappa}_{AB}$  gives rise to a Killing vector  $\tilde{\xi}_a$  whose spinorial counterpart is

$$\tilde{\xi}_{AA'} \equiv \tilde{\nabla}^B{}_{A'} \tilde{\kappa}_{AB}. \quad (25)$$

Furthermore, if  $\tilde{\xi}_a$  is a Killing vector in the physical spacetime then,

$$X_a \equiv \Xi^2 \tilde{\xi}_a \quad (26)$$

corresponds to a conformal Killing vector for the unphysical spacetime, namely, it can be verified that

$$\nabla_a X_b + \nabla_b X_a = \frac{1}{2} \nabla^a X_a g_{ab}. \quad (27)$$

This conformal Killing vector additionally satisfy

$$X^a \nabla_a \Xi = \frac{1}{4} \nabla_a X^a. \quad (28)$$

Equations (27)-(28) are the so-called *unphysical Killing equations* —see [23] for a discussion on the unphysical Killing equations. A direct computation shows that given a Killing spinor in the unphysical spacetime  $\kappa_{AB}$ , the concomitant

$$X_{AA'} \equiv \Xi \xi_{AA'} - 3\kappa_{AQ} \nabla_{A'}{}^Q \Xi \quad (29)$$

corresponds to the spinorial counterpart of a conformal Killing vector satisfying equations (27) and (28). However, in general, the spinor  $\xi_{AA'}$  has no straightforward interpretation and will be regarded as an auxiliary variable for the subsequent discussion. Additionally, observe that the introduction of  $\xi_{AA'}$  allows one to write the irreducible decomposition of the gradient of the Killing spinor as

$$\nabla_{AA'} \kappa_{BC} = \frac{1}{3} H_{A'ABC} - \frac{1}{3} \xi_{CA'} \epsilon_{AB} - \frac{1}{3} \xi_{BA'} \epsilon_{AC}. \quad (30)$$

On the other hand, a direct computation using the definition of  $H_{A'ABC}$  shows that the zero-quantity  $B_{ABCD}$  encodes the Buchdahl constraint, namely

$$B_{ABCD} = \kappa_{(D}{}^F \Psi_{ABC)F}. \quad (31)$$

To complete the discussion observe that if the Killing spinor equation  $H_{A'ABC} = 0$  is satisfied, the Killing spinor  $\kappa_{AB}$  and the auxiliary spinor  $\xi_{AA'}$  satisfy the following wave equations:

$$\square \kappa_{BC} = -4\Lambda \kappa_{BC} + \kappa^{AD} \Psi_{BCAD}, \quad (32)$$

$$\begin{aligned} \square \xi_{AA'} = & -\frac{4}{3} \Delta^{BC}{}_{BA'} \kappa_{AC} - 2\Lambda_{A'}{}^A{}^{BC} \kappa_{BC} - 6\xi_{AA'} \Lambda - 2\xi^{BB'} \Phi_{ABA'B'} \\ & - \frac{3}{2} \kappa^{BC} Y_{ABCA'} + \Psi_{ABCD} H_{A'}{}^{BCD} - 12\kappa_{AB} \nabla^B{}_{A'} \Lambda - \frac{1}{2} \kappa^{BC} \Delta_{(ABC)A'} \end{aligned} \quad (33)$$

The wave equation (32) is derived considering the integrability condition  $\nabla^{AA'} H_{A'ABC} = 0$ , substituting equation (22a) and exploiting the spinorial Ricci identities (3a)-(3b). To obtain equation (33) observe that from equation (23) one has

$$\square \xi_{AA'} = \nabla_{CC'} \nabla^{CC'} \nabla^B{}_{A'} \kappa_{AB}. \quad (34)$$

Commuting covariant derivatives in the last expression renders

$$\square \xi_{AA'} = \square_{A'B'} \nabla^{BB'} \kappa_{AB} + \square^C{}_B \nabla^B{}_{A'} \kappa_{AC} - \nabla_{BA'} \square^{CD} \kappa_{AC} + \nabla^B{}_{A'} \square \kappa_{AB} - \nabla^B{}_{B'} \square_{A'}{}^{B'} \kappa_{AB}.$$

Then, a lengthy computation using the decomposition of the auxiliary vector as given in equation (30), the Bianchi identities (21a)-(21b) and the spinorial Ricci identities (3a)-(3b) render equation (33). The above discussion is summarised in the following

**Lemma 1.** *Let  $(\mathcal{M}, g)$  represent a solution to the conformal Einstein field equations admitting a Killing spinor  $\kappa_{AB}$ ; namely suppose that*

$$H_{A'ABC} = 0, \quad Z_{AA'BB'} = 0, \quad Z_{AA'} = 0, \quad \Delta_{ABCC'} = 0, \quad \Pi_{AA'BB'} = 0, \quad \Lambda_{AB'BC} = 0.$$

*Let  $\xi_{AA'}$  denote the auxiliary vector defined as in equation (23), then*

$$\nabla_{CC'}\xi_{DD'} + \nabla_{DD'}\xi_{CC'} + 6\kappa_{(D}{}^A\Phi_{C)AC'D'} = 0, \quad \kappa_{(D}{}^F\Psi_{ABC)F} = 0.$$

*Moreover*

$$X_{AA'} = \Xi\xi_{AA'} - 3\kappa_{AQ}\nabla_{A'}{}^Q\Xi$$

*is the spinorial counterpart of a conformal Killing vector  $X_a$  satisfying the unphysical Killing vector equations (27)–(28). In addition, the Killing spinor  $\kappa_{AB}$  and the auxiliary vector  $\xi_{AA'}$  satisfy the following wave equations*

$$\square\kappa_{BC} = -4\Lambda\kappa_{BC} + \kappa^{AD}\Psi_{BCAD}, \quad (35)$$

$$\square\xi_{AA'} = -6\xi_{AA'}\Lambda - 2\xi^{BB'}\Phi_{ABA'B'} - \frac{3}{2}\kappa^{BC}Y_{ABCA'} - 12\kappa_{AB}\nabla^B{}_{A'}\Lambda. \quad (36)$$

In the following, we aim to identify the initial data for  $\kappa_{AB}$  which, when propagated according to (35) gives rise to a Killing spinor on the spacetime development. It is important to note that, since  $\kappa_{AB}$  solves equation (35) by construction, equation (35) can be assumed to hold throughout  $\mathcal{M}$ .

## 5 Propagation equations

In this section we construct, given a solution to equations (35)–(36) on  $\mathcal{M}$ , a set of wave equations for the zero-quantities  $H_{A'ABC}$  and  $S_{AA'BB'}$  which are homogeneous in these zero-quantities and their first derivatives.

### 5.1 The general strategy

As discussed in detail in Sections 5.3–5.5, deriving the required wave equation for  $S_{A'ABC}$  is more involved than the one for  $H_{A'ABC}$ . In order to obtain an homogeneous wave equation for  $S_{AA'BB'}$ , we first derive separately two inhomogeneous equations, which when combined yield the desired homogeneous wave equation. The first inhomogeneous equation, derived in Section 5.3, makes use of the definition of  $S_{AA'BB'}$  in terms of the auxiliary vector (24). The second inhomogeneous equation, derived in Section 5.5, is obtained through the use of equations (22b)–(22c) and (31). The procedure is summarised in the schematic of Figure 1.

Once the desired wave equations are obtained the following result for homogeneous wave equations will be used:

**Theorem 1.** *Let  $\mathcal{M}$  be a smooth manifold equipped with a Lorentzian metric  $g$  and consider the wave equation*

$$\square\underline{u} = h(\underline{u}, \partial\underline{u})$$

*where  $\underline{u} \in \mathbb{C}^m$  is a complex vector-valued function on  $\mathcal{M}$ ,  $h : \mathbb{C}^{2m} \rightarrow \mathbb{C}^m$  is a smooth homogeneous function of its arguments and  $\square = g^{ab}\nabla_a\nabla_b$ . Let  $\mathcal{U} \subset \mathcal{S}$  be an open set and  $\mathcal{S} \subset \mathcal{M}$  be a spacelike hypersurface with normal  $\tau^a$  respect to  $g$ . Then the Cauchy problem*

$$\begin{aligned} \square\underline{u} &= h(\underline{u}, \partial\underline{u}), \\ \underline{u}|_{\mathcal{U}} &= \underline{u}_0, \quad \mathcal{P}\underline{u}|_{\mathcal{U}} = \underline{u}_1, \end{aligned}$$

*where  $\underline{u}_0$  and  $\underline{u}_1$  are smooth on  $\mathcal{U}$  and  $\mathcal{P} \equiv \tau^\mu\nabla_\mu$ , has a unique solution  $\underline{u}$  in the domain of dependence of  $\mathcal{U}$ .*

We refer the reader to [32, 31] for a proof —see also Theorem 1 in [16].

$$\left. \begin{aligned}
\mathbf{S} &= \Phi \times \kappa + \nabla \times \xi \longrightarrow \square \mathbf{S} = \mathbf{I}_{sym} + \mathbf{h}_1 \\
\nabla \times \mathbf{H} &= \mathbf{B} + \mathbf{S} \longrightarrow (\nabla \times \nabla \times \mathbf{B})_{sym} = \square \mathbf{S}_{sym} + \mathbf{h}_{2\ sym} \\
\mathbf{B} &= \Psi \times \kappa \longrightarrow (\nabla \times \nabla \times \mathbf{B})_{sym} = \mathbf{E}_{sym} + \mathbf{h}_{3\ sym}
\end{aligned} \right\} \square \mathbf{S} = \mathbf{h}$$

Figure 1: Schematic description of the derivation of the wave equation for  $S_{AA'BB'}$ , given in Sections 5.3-5.5. In this diagram spinors are represented simply by their kernel letters, e.g.,  $T_{AB\dots F,C'D'\dots H'}$  is denoted by  $\mathbf{T}$ . In addition, the symbol  $\times$  has been used to denote, in a schematic way, contractions between spinors. The subscript *sym* has been added to indicate that a given expression is symmetric. The quantities  $\mathbf{h}_1$ ,  $\mathbf{h}_2$  and  $\mathbf{h}_3$  denote homogeneous expressions in  $\mathbf{H}$ ,  $\nabla \times \mathbf{H}$ ,  $\mathbf{S}$  and  $\nabla \times \mathbf{S}$ . The quantities  $\mathbf{I}_{sym}$  and  $\mathbf{E}_{sym}$  encode the inhomogeneous terms appearing in the corresponding equations.

**Remark 3.** Recall that an equation of the above form are said to be *homogeneous in  $\underline{u}$  and its first derivatives* if  $h(\lambda \underline{u}, \lambda \partial \underline{u}) = \lambda h(\underline{u}, \partial \underline{u})$  for all  $\lambda \in \mathbb{C}$ .

Observe that it follows from the uniqueness property of Theorem 1 that the zero-quantities are *propagated*; that is to say that if the initial conditions

$$\begin{aligned}
H_{A'ABC} &= 0, \\
\mathcal{P}H_{A'ABC} &= 0, \\
S_{AA'BB'} &= 0, \\
\mathcal{P}S_{AA'BB'} &= 0,
\end{aligned}$$

hold on  $\mathcal{U} \subset \mathcal{S}$ , then  $H_{A'ABC}$  and  $S_{AA'BB'}$  vanish identically on the domain of dependence of  $\mathcal{U}$ . The above initial conditions may then be translated into necessary and sufficient conditions for the Killing spinor candidate,  $\kappa_{AB}$ , restricted to a Cauchy hypersurface  $\mathcal{S}$ ; this is done in Section 6.

## 5.2 Wave equation for $H_{A'ABC}$

To construct the wave equation for  $H_{A'ABC}$  one starts from the identity

$$\square H_{A'ABC} = \nabla_{DD'} \nabla_A^{D'} H_{A'BC}^D - \nabla_{AD'} \nabla_D^{D'} H_{A'}^D{}_{BC}.$$

Substituting the definition of  $S_{AA'BB'}$ , as given in equation (22b), in the second term of the last expression and using equations (3a)-(3b) one obtains

$$\square H_{A'ABC} = 10\Lambda H_{A'ABC} - 4\Psi_{ADF(B} H_{|A'|C)}^{DF} - 2\Phi_A^D{}_{A'}^{D'} H_{D'BCD} - 2\nabla_{AD'} S_B^{D'}{}_{CA'}. \quad (37)$$

Exploiting the symmetries of  $H_{A'ABC}$  one obtains

$$\square H_{A'ABC} = 10\Lambda H_{A'ABC} + 2\nabla_{(A}^{D'} S_{B|D'|C)A'} - 2\Phi_{(A}^D{}_{|A'}^{D'} H_{D'|BC)D} - 4\Psi_{(AB}^{DF} H_{|A'|C)DF}. \quad (38)$$

Observe that equations (38) and (37) contain the same information as the traces of equation (37) represent identities which follow from the definition of  $S_{AA'BB'}$  in terms of  $H_{A'ABC}$  as given in equation (22b). These identities will be useful for the subsequent discussion.

$$\nabla_{AD'} S^{AD'}{}_{CA'} = -\Psi_{CADB} H_{A'}^{ADB} + \Phi^{AD}{}_{A'}^{D'} H_{D'CAD}, \quad (39a)$$

$$\nabla_{AD'} S_B^{D'A}{}_{A'} = -\Psi_{BADC} H_{A'}^{ADC} + \Phi^{AD}{}_{A'}^{D'} H_{D'BAD}, \quad (39b)$$

$$\nabla_{AD'} S^{BD'}{}_{BA'} = 0. \quad (39c)$$

Notice that equation (38) is a wave equation homogeneous in  $H_{A'ABCD}$ ,  $S_{AA'BB'}$  and their first derivatives.

### 5.3 Inhomogeneous wave equation for $S_{AA'BB'}$

The purpose of this section is to derive the first of two inhomogeneous wave equations for  $S_{AA'BB'}$ , which, when combined, will yield the desired homogeneous equation. To proceed, some ancillary spinorial decompositions will be required.

#### 5.3.1 Ancillary decompositions

This section collects some ancillary decompositions which will prove useful for the derivation of the wave equation for  $S_{AA'BB'}$ . To start the discussion, observe that from expression (22b) it follows that

$$\nabla_{CC'} \xi^{CC'} = -\frac{1}{2} S_{CC'}{}^{CC'}, \quad \nabla_{(A|(A' \xi_{B')|B)} = -\frac{1}{2} S_{(AB)(A'B')} - 3\kappa_{(A}{}^C \Phi_{B)CA'B'}. \quad (40)$$

Using the above expressions the gradient of the auxiliary vector  $\xi_{AA'}$  can be decomposed as

$$\begin{aligned} \nabla_{AA'} \xi_{BB'} = & -3\kappa_{(B}{}^C \Phi_{A)CA'C'} - \frac{1}{8} S^{CC'}{}_{CC'} \epsilon_{AB} \epsilon_{A'B'} - \frac{1}{2} S_{(AB)(A'B')} \\ & - \frac{1}{2} \epsilon_{A'B'} \nabla_{(A}{}^{C'} \xi_{B)C'} - \frac{1}{2} \epsilon_{AB} \nabla^C{}_{(A'} \xi_{B')C}. \end{aligned} \quad (41)$$

Using the definitions of  $H_{A'ABC}$  and  $\xi_{AA'}$  encoded in equations (22a) and (23) respectively, one can reexpress the gradient of the Killing spinor as

$$\nabla_{AA'} \kappa_{BC} = \frac{1}{3} H_{A'ABC} - \frac{1}{3} \xi_{CA'} \epsilon_{AB} - \frac{1}{3} \xi_{BA'} \epsilon_{AC}. \quad (42)$$

The irreducible decomposition of the gradient of the trace-free Ricci spinor the second Bianchi identity as expressed in (21a) and the equation encoded in the zero-quantity (20c) implies

$$\begin{aligned} \nabla_{AA'} \Phi_{BCB'C'} = & \frac{1}{6} \bar{Y}_{A'B'C'C} \epsilon_{AB} + \frac{1}{6} \bar{Y}_{A'B'C'B} \epsilon_{AC} + \frac{1}{6} Y_{ABCC'} \bar{\epsilon}_{A'B'} + \frac{1}{6} Y_{ABCB'} \bar{\epsilon}_{A'C'} \\ & - \frac{2}{3} \bar{\epsilon}_{A'C'} \epsilon_{A(C} \nabla_{B)} \Lambda - \frac{2}{3} \bar{\epsilon}_{A'B'} \epsilon_{A(C} \nabla_{B)} \Lambda + \nabla_{(A'} (A \Phi_{BC})_{|B'C'}). \end{aligned} \quad (43)$$

Another identity that will be used involving second derivatives of the tracefree Ricci spinor is derived as follows: Applying  $\nabla^A{}_{B'}$  to the zero-quantity encoded in (20c), and after a lengthy computation using equations (21a)-(21b) and (20a)-(20e), one obtains the identity

$$\begin{aligned} \square \Phi_{BCC'D'} = & 8\Lambda \Phi_{BCB'C'} - 2 \Phi_B{}^A{}_{C'}{}^{A'} \Phi_{CAB'A'} - 2 \Phi_B{}^A{}_{B'}{}^{A'} \Phi_{CAC'A'} - 2 \Phi_{BC}{}^{A'D'} \Psi_{B'C'A'D'} \\ & + \frac{3}{2} \epsilon_{BC} \bar{\epsilon}_{B'C'} \square \Lambda - 2 \nabla_{(B} (B' \nabla_{C')} \Lambda + \nabla_{AB'} Y_B{}^{A'A}{}_{A'CC'} + 2 \nabla_{AB'} \nabla_{BA'} \Phi_C{}^A{}_{C'}{}^{A'}. \end{aligned} \quad (44)$$

Additionally, the following decompositions will be used in the subsequent discussion

$$\nabla_{AA'} \nabla_{BB'} \Lambda = \frac{1}{4} \epsilon_{AB} \bar{\epsilon}_{A'B'} \square \Lambda + \nabla_{(A'} (A \nabla_{B')} \Lambda, \quad (45)$$

$$\begin{aligned} \xi_{DD'} \nabla_{CC'} \Lambda = & \frac{1}{4} \xi^{AA'} \epsilon_{CD} \bar{\epsilon}_{C'D'} \nabla_{AA'} \Lambda + \xi_{(C} (C' \nabla_{D')} \Lambda \\ & + \frac{1}{2} \bar{\epsilon}_{C'D'} \xi_{(C}{}^{A'} \nabla_{D)A'} \Lambda + \frac{1}{2} \epsilon_{CD} \xi^A{}_{(C'} \nabla_{|A|D')} \Lambda. \end{aligned} \quad (46)$$

#### 5.3.2 Wave equation for $S_{AA'BB'}$

Applying  $\nabla_{PP'} \nabla^{PP'}$  to the expression for  $S_{AA'BB'}$  in terms of first derivatives of the auxiliary vector as encoded in equation (24) one obtains

$$\square S_{CC'DD'} = -6 \nabla_{PP'} \nabla^{PP'} (\kappa_{(C}{}^Q \Phi_{D)QC'D'}) - \nabla_{PP'} \nabla^{PP'} \nabla_{CC'} \xi_{DD'} - \nabla_{PP'} \nabla^{PP'} \nabla_{DD'} \xi_{CC'}. \quad (47)$$

Commuting covariant derivatives renders

$$\begin{aligned}\square S_{CC'DD'} &= -6\Phi_{C'D'A(C}\square\kappa_{D)}^A - 6\kappa_{(C}^A\square\Phi_{D)AC'D'} - \nabla_{CC'}\square\xi_{DD'} - \nabla_{DD'}\square\xi_{CC'} \\ &\quad - \square_{CA}\nabla^A_{C'}\xi_{DD'} - \square_{C'A'}\nabla_{C'}^A\xi_{DD'} - \square_{DA}\nabla^A_{D'}\xi_{CC'} - \square_{D'A'}\nabla_{D'}^A\xi_{CC'} \\ &\quad + \nabla_{AC'}\square_{C'}^A\xi_{DD'} + \nabla_{AD'}\square_D^A\xi_{CC'} + \nabla_{CA'}\square_{C'}^A\xi_{DD'} + \nabla_{DA'}\square_{D'}^A\xi_{CC'} \\ &\quad - 6\nabla_{BA'}\Phi_{DAC'D'}\nabla^{BA'}\kappa_C^A - 6\nabla_{BA'}\Phi_{CAC'D'}\nabla^{BA'}\kappa_D^A.\end{aligned}$$

At this point one can substitute the wave equation for the Killing spinor, the auxiliary spinor and the tracefree Ricci spinor as given in equations (35), (36) and (44) respectively. Additionally, observe that the the spinorial Ricci identities (3a)-(3b) can be employed to replace the operator  $\square_{AB}$  by curvature terms. Substituting equations (35), (36), (44), (31), the conformal Einstein field equations as encoded in (19) and the auxiliary results collected in Section 5.3.1, one obtains the following wave equation for  $S_{CC'DD'}$

$$\begin{aligned}\square S_{CC'DD'} &= 12B_{CDAB}\Phi^{AB}_{C'D'} - 3\xi^A_{D'}Y_{CDAC'} - 3\xi^A_{C'}Y_{CDAD'} \\ &\quad + 3\kappa_D^A\nabla_{BC'}Y_{CA}^B{}_{D'} + 3\kappa_C^A\nabla_{BC'}Y_{DA}^B{}_{D'} + \frac{3}{2}\kappa^{AB}\nabla_{DD'}Y_{CABC'} + \frac{3}{2}\kappa^{AB}\nabla_{CC'}Y_{DABD'} \\ &\quad + \frac{1}{6}Y_D^{AB}{}_{D'}H_{C' CAB} + \frac{1}{6}Y_C^{AB}{}_{C'}H_{D'DAB} - \frac{1}{3}Y_C^{AB}{}_{D'}H_{C'DAB} - \frac{1}{3}Y_D^{AB}{}_{C'}H_{D' CAB} \\ &\quad + \frac{8}{3}H_{D' CDA}\nabla^A_{C'}\Lambda + \frac{8}{3}H_{C' CDA}\nabla^A_{D'}\Lambda - \frac{2}{3}\bar{Y}_{C'D'}^{A'A}H_{A'CDA} - 3\Lambda S^{AA'}_{AA'}\epsilon_{CD}\bar{\epsilon}_{C'D'} \\ &\quad - \nabla_{CC'}(H_{D'}^{ABF}\Psi_{DABF}) - \nabla_{DD'}(H_{C'}^{ABF}\Psi_{CABF}) \\ &\quad - 2\Psi_{CDAB}S^{(A}_{(C'}{}^{B)}_{D')} - 2\bar{\Psi}_{C'D'A'B'}S_{(C}^{(A'}{}^{B')}_{D)} + 4\Lambda S_{(C|(C'|D)D')} \\ &\quad - 2\Phi_D^A{}_{D'}{}^{A'}(S_{(C|(C'|A)A'}) - 2\Phi_D^A{}_{C'}{}^{A'}S_{(C|(D'|A)A')} - 2\Phi_C^A{}_{D'}{}^{A'}S_{(D|(C'|A)A')} \\ &\quad - 2\Phi_C^A{}_{C'}{}^{A'}S_{(D|(D'|A)A')} - 2H^{A'}_D{}^{AB}\nabla_{C'}(C\Phi_{AB})|D'A') \\ &\quad - 2H^{A'}_C{}^{AB}\nabla_{(C'}(D\Phi_{AB})|D'A').\end{aligned}\tag{48}$$

Observe that the latter is an homogeneous expression in  $H_{A'ABC}$ ,  $S_{AA'BB'}$  and its first derivatives except for the term

$$\begin{aligned}I_{CC'DD'} &\equiv 12B_{CDAB}\Phi^{AB}_{C'D'} - 3\xi^A_{D'}Y_{CDAC'} - 3\xi^A_{C'}Y_{CDAD'} + 3\kappa_D^A\nabla_{BC'}Y_{CA}^B{}_{D'} \\ &\quad + 3\kappa_C^A\nabla_{BC'}Y_{DA}^B{}_{D'} + \frac{3}{2}\kappa^{AB}\nabla_{CC'}Y_{DABD'} + \frac{3}{2}\kappa^{AB}\nabla_{DD'}Y_{CABC'}.\end{aligned}$$

As an additional remark observe that taking a trace of equation (48) one obtains

$$\begin{aligned}\square S_{CC'}{}^C{}_{D'} &= \frac{1}{2}\bar{Y}_{D'}^{A'B'A}\bar{H}_{AC'A'B'} - \frac{1}{2}\bar{Y}_{C'}^{A'B'A}\bar{H}_{AD'A'B'} - 6\Lambda\bar{S}^{A'A}{}_{A'A}\bar{\epsilon}_{C'D'} \\ &\quad - \nabla_{AC'}(\bar{H}^{AA'B'F'}\bar{\Psi}_{D'A'B'F'}) + \nabla_{AD'}(\bar{H}^{AA'B'F'}\bar{\Psi}_{C'A'B'F'}).\end{aligned}\tag{49}$$

Observe that the right-hand side of the last equation is an homogeneous expression in  $H_{A'ABC}$ ,  $S_{AA'BB'}$  and its first derivatives. Consequently, exploiting the irreducible decomposition of  $S_{AA'BB'}$  to write

$$\begin{aligned}\square S_{CC'DD'} &= \square S_{(CD)(C'D')} - \frac{1}{2}\bar{\epsilon}_{C'D'}\square S_{(C}^{A'}{}_{D)A'} \\ &\quad - \frac{1}{2}\epsilon_{CD}\square S^A_{(C'|A|D')} + \frac{1}{4}\epsilon_{CD}\bar{\epsilon}_{C'D'}\square S^{AA'}_{AA'},\end{aligned}\tag{50}$$

one can re-express equation (48) as

$$\square S_{CC'DD'} = I_{(CD)(C'D')} + F_{CC'DD'},\tag{51}$$

where  $F_{CC'DD'}$  is an homogeneous expression depending on  $H_{A'ABC}$ ,  $S_{AA'BB'}$  and its first derivatives. Notice that the inhomogeneous term  $I_{(CD)(C'D')}$  contains the Buchdahl constraint  $B_{ABCD}$ . Consequently, one needs to analyse more closely this quantity. An immediate but important observation for the latter discussion is that the main obstruction to obtaining an homogeneous wave equation is contained in the symmetric part of  $S_{CDC'D'}$ :

$$\square S_{(CD)(C'D')} = I_{(CD)(C'D')} + F_{(CD)(C'D')}.\tag{52}$$

## 5.4 Derivatives of the Buchdahl zero-quantity

The presence of the Buchdahl zero-quantity in the inhomogeneous term of equation (52) suggests that it is necessary to find auxiliary identities associated to the Buchdahl constraint. The aim of this section is to derive such identities by expressing  $\nabla^C_{C'}\nabla^A_{A'}B_{ABCD}$  in two different ways by exploiting equations (22b)-(22c) and (31).

### 5.4.1 First approach to express second derivatives of the Buchdahl constraint

The irreducible decomposition of  $\nabla_{AA'}H^{A'}_{BCD}$  and definitions (22b)-(22c) give

$$\nabla_{AA'}H^{A'}_{BCD} = -6B_{ABCD} - \frac{3}{4}\epsilon_{A(B}S_{C}^{A'}{}_{D)A'}. \quad (53)$$

Applying  $\nabla^A_{Q'}$  to the last expression one obtains

$$\nabla^A_{Q'}\nabla_{AA'}H^{A'}_{BCD} = 6\nabla_{AQ'}B_{BCD}{}^A - \frac{3}{4}\nabla_{Q'(B}S_{C}^{A'}{}_{D)A'}. \quad (54)$$

Exploiting the spinorial Ricci identities (3a)-(3b) in equation (54) and rearranging one derives the following expression

$$\begin{aligned} \nabla_{AQ'}B_{BCD}{}^A &= \frac{1}{12}\square H_{Q'BCD} - \frac{1}{2}\Lambda H_{Q'BCD} \\ &+ \frac{1}{6}\Phi_D{}^A{}_{Q'}{}^{A'}H_{A'BCA} + \frac{1}{6}\Phi_C{}^A{}_{Q'}{}^{A'}H_{A'BDA} + \frac{1}{6}\Phi_B{}^A{}_{Q'}{}^{A'}H_{A'CDA} \\ &+ \frac{1}{24}\nabla_{BQ'}S_C{}^{A'}{}_{DA'} + \frac{1}{24}\nabla_{CQ'}S_B{}^{A'}{}_{DA'} + \frac{1}{24}\nabla_{DQ'}S_B{}^{A'}{}_{CA'}. \end{aligned}$$

At this point we can substitute for the D'Alembertian of the zero quantity  $H_{A'ABC}$  using the wave equation (38). Having done so, one can apply a further derivative to the above equation to obtain

$$\nabla^C_{C'}\nabla^A_{A'}B_{ABCD} = \frac{1}{36}\square S_{(B|C'|D)A'} - \frac{1}{48}\square S_B{}^{B'}{}_{DB'}\bar{\epsilon}_{A'C'} + \Sigma_{C'A'BD} + P_{C'A'BD} \quad (55)$$

where  $P_{C'A'BD}$  is an homogeneous expression in on  $H_{A'ABC}$ ,  $S_{AA'BB'}$  and its first derivatives, given in appendix 8, and where  $\Sigma_{C'A'BD}$  is given by

$$\begin{aligned} \Sigma_{C'A'BD} &\equiv \frac{1}{24}\nabla_{AC'}\nabla_{BA'}S^{AB'}{}_{DB'} - \frac{1}{36}\nabla_{AC'}\nabla_{BB'}S^{AB'}{}_{DA'} - \frac{1}{36}\nabla_{AC'}\nabla_{BB'}S_D{}^{B'}{}_{A'} \\ &+ \frac{1}{24}\nabla_{AC'}\nabla_{DA'}S_B{}^{B'}{}_{A'} - \frac{1}{36}\nabla_{AC'}\nabla_{DB'}S^{AB'}{}_{BA'} - \frac{1}{36}\nabla_{AC'}\nabla_{DB'}S_B{}^{B'}{}_{A'}. \end{aligned}$$

Commuting covariant derivatives and using the identities (39a)-(39c) one can rewrite the above expression as follows

$$\Sigma_{C'A'BD} = \frac{1}{18}\square S_{(B|C'|D)A'} + Q_{C'A'BD} \quad (56)$$

where  $Q_{C'A'BD}$  is an homogeneous expression in on  $H_{A'ABC}$ ,  $S_{AA'BB'}$  and its first derivatives, given in Appendix 8. Consequently, one has

$$\nabla^C_{C'}\nabla^A_{A'}B_{ABCD} = \frac{1}{12}\square S_{(B|C'|D)A'} - \frac{1}{48}\square S_B{}^{B'}{}_{DB'}\bar{\epsilon}_{A'C'} + P_{C'A'BD} + Q_{C'A'BD} \quad (57)$$

### 5.4.2 Second approach to express second derivatives of the Buchdahl constraint

An alternative way to obtain an expression for  $\nabla^C_{C'}\nabla^A_{A'}B_{ABCD}$  is to start from equation (31) to obtain

$$\nabla^A_{A'}B_{ABCD} = \Psi_{F(BCD}\nabla^A_{|A'|\kappa_A)}{}^F + \kappa_{(D}{}^F\nabla^A_{|A'|\Psi_{ABC)F}}.$$

Exploiting the decomposition (42) and the conformal Einstein field equations as encoded in (19) gives

$$\nabla^A_{A'}B_{ABCD} = \frac{1}{2}\xi^A{}_{A'}\Psi_{BCDA} - \frac{3}{8}\kappa_{(B}{}^AY_{CD)AA'} + \frac{1}{4}\Psi_{AF(BC}H_{D)A'}{}^{AF} - \frac{1}{4}\kappa^{AF}\nabla_{FA'}\Psi_{BCDA}. \quad (58)$$

Applying  $\nabla^C{}_{C'}$  to equation (58), a long calculation exploiting the irreducible decomposition of  $\nabla_{AA'}\kappa_{BC}$  and  $\nabla_{AA'}\xi_{BB'}$ , commuting covariant derivatives and using the conformal Einstein field equations as given in (19) renders

$$\begin{aligned}\nabla^C{}_{C'}\nabla^A{}_{A'}B_{ABCD} = & \frac{1}{4}(\Phi_D{}^F{}_{A'C'}\Psi_{BACF} - 4\Phi_A{}^F{}_{A'C'}\Psi_{BD CF} + \Phi_B{}^F{}_{A'C'}\Psi_{DACF})\kappa^{AC} \\ & + \frac{1}{4}\bar{\epsilon}_{A'C'}(\Psi_{ACFG}\Psi_{BD}{}^{FG} + 2\Psi_{BA}{}^{FG}\Psi_{DCFG} - 6\Lambda\Psi_{BDAC})\kappa^{AC} \\ & + \frac{1}{8}\kappa^{AC}\nabla_{CA'}Y_{BDAC'} + \frac{1}{8}\kappa_D{}^A\nabla_{CC'}Y_{BA}{}^C{}_{A'} - \frac{1}{4}\xi^A{}_{C'}Y_{BDAA'} \\ & + \frac{1}{8}\kappa^{AC}\nabla_{CC'}Y_{BDAA'} + \frac{1}{8}\kappa_B{}^A\nabla_{CC'}Y_{DA}{}^C{}_{A'} - \frac{1}{4}\xi^A{}_{A'}Y_{BDAC'} \\ & + \frac{1}{4}\Psi_{BDAC}\bar{\epsilon}_{A'C'}\nabla^{(A|B'|}\xi^{C)}{}_{B'} + U_{A'BC'D},\end{aligned}\quad (59)$$

where  $U_{A'BC'D}$  is an homogeneous expression in on  $H_{A'ABC}$ ,  $S_{AA'BB'}$  and its first derivatives, given in Appendix 8.

### 5.5 Wave equation for $S_{(AB)(A'B')}$

In Section 5.4 two different expressions for  $\nabla^C{}_{C'}\nabla^A{}_{A'}B_{ABCD}$  were computed. Observe that the right-hand side of equation (57) contains  $\square S_{(B|C'|D)A'}$  while (59) does not. Consequently, one can use equations (57) and (59) to obtain a wave equation for  $S_{(B|C'|D)A'}$ . A direct computation using equations (57), (59) and (31) renders

$$\square S_{(AB)(A'B')} = \mathcal{I}_{ABA'B'} + W_{ABA'B'}, \quad (60)$$

where  $W_{ABA'B'} = W_{(AB)(A'B')}$  and  $\mathcal{I}_{ABA'B'} = \mathcal{I}_{(AB)(A'B')}$  are homogeneous expressions in  $H_{A'ABC}$ ,  $S_{AA'BB'}$  and their first derivatives, given by

$$\begin{aligned}W_{ABA'B'} \equiv & 12(U_{(AB)(A'B')} - P_{(AB)(A'B')} - Q_{(AB)(A'B')}), \\ \mathcal{I}_{ABA'B'} \equiv & -12B_{ABCD}\Phi^{CD}{}_{A'B'} + 3\xi^C{}_{B'}Y_{ABCA'} + 3\xi^C{}_{A'}Y_{ABCB'} \\ & - \frac{3}{4}\kappa_B{}^C\nabla_{DA'}Y_{AC}{}^D{}_{B'} - \frac{3}{4}\kappa_A{}^C\nabla_{DA'}Y_{BC}{}^D{}_{B'} - \frac{3}{2}\kappa^{CD}\nabla_{DB'}Y_{ABCA'} \\ & - \frac{3}{4}\kappa_B{}^C\nabla_{DB'}Y_{AC}{}^D{}_{A'} - \frac{3}{4}\kappa_A{}^C\nabla_{DB'}Y_{BC}{}^D{}_{A'} - \frac{3}{2}\kappa^{CD}\nabla_{DA'}Y_{ABCB'}.\end{aligned}$$

Through use of the following identity

$$\kappa^{CD}\nabla_{AA'}Y_{BCDB'} = \kappa^{CD}\nabla_{DA'}Y_{ABCB'} - \kappa_A{}^C\nabla_{DA'}Y_{BC}{}^D{}_{B'}$$

it may be shown that in fact

$$\mathcal{I}_{ABA'B'} = -\frac{1}{2}I_{(AB)(A'B')}.$$

### 5.6 Homogeneous wave equation for $S_{AA'BB'}$

As discussed before, the main obstruction to obtaining an homogeneous wave equation is contained to the symmetric part of  $S_{ABA'B'}$ . More specifically, the obstruction is contained in the terms denoted by  $I_{AA'BB'}$  and  $\mathcal{I}_{AA'BB'}$  in equations (52) and (60) respectively. However, one can take linear combinations of equations (52) and (60) to remove the inhomogeneous terms. In particular one has

$$3\square S_{(AB)(A'B')} = I_{(AB)(A'B')} + 2\mathcal{I}_{ABA'B'} + F_{(AB)(A'B')} + 2W_{ABA'B'}.$$

After a direct computation using the explicit form of  $I_{ABA'B'}$  and  $\mathcal{I}_{ABA'B'}$  one concludes that

$$\square S_{(AB)(A'B')} = \frac{1}{3}F_{(AB)(A'B')} + \frac{2}{3}W_{ABA'B'}. \quad (61)$$

Finally, using equation (61), (50) and (51) one obtains

$$\square S_{AA'BB'} = \frac{1}{3}F_{(AB)(A'B')} - \frac{1}{2}\bar{\epsilon}_{A'B'}F_{(A}{}^{Q'}{}_{B)Q'} - \frac{1}{2}\epsilon_{AB}F^{Q'}{}_{(A'|Q|B')}$$

$$+ \frac{1}{4} \epsilon_{AB} \bar{\epsilon}_{A'B'} F^{QQ'}{}_{QQ'} + \frac{2}{3} W_{ABA'B'}. \quad (62)$$

Notice that latter encodes an homogeneous expressions for  $S_{AA'BB'}$  as  $F_{(AB)(A'B')}$  and  $W_{ABA'B'}$  represent homogeneous expressions on  $H_{A'ABC}$ ,  $S_{AA'BB'}$  and its first derivatives.

We are now in a position to state the following proposition:

**Proposition 1.** *Given initial data for the alternative conformal field equations on a spacelike hypersurface  $\mathcal{S}$  with normal vector  $\tau^{AA'}$ , and associated normal derivative  $\mathcal{P} \equiv \tau^{AA'} \nabla_{AA'}$ , the corresponding spacetime development admits a Killing spinor in the domain of dependence of  $\mathcal{U} \subset \mathcal{S}$  if and only if*

$$H_{A'ABC} = 0, \quad (63a)$$

$$\mathcal{P}H_{A'ABC} = 0, \quad (63b)$$

$$S_{AA'BB'} = 0, \quad (63c)$$

$$\mathcal{P}S_{AA'BB'} = 0 \quad (63d)$$

hold on  $\mathcal{U}$ .

*Proof.* The *only if* direction is immediate. Suppose, on the other hand, that (63a)–(63d) hold on some  $\mathcal{U} \subset \mathcal{S}$  —that is to say, there exist spinor fields  $\kappa_{AB}$ ,  $\xi_{AA'}$  for which (63a)–(63d) are satisfied on  $\mathcal{U}$ . The latter is then used as initial data for the wave equations

$$\begin{aligned} \square \kappa_{BC} &= -4\Lambda \kappa_{BC} + \kappa^{AD} \Psi_{BCAD}, \\ \square \xi_{AA'} &= -6\xi_{AA'} \Lambda - 2\xi^{BB'} \Phi_{ABA'B'} - \frac{3}{2} \kappa^{BC} Y_{ABCA'} - 12\kappa_{AB} \nabla^B{}_{A'} \Lambda. \end{aligned}$$

As the zero-quantities  $H_{A'ABC}$ ,  $S_{AA'BB'}$  satisfy the homogeneous wave equations (38), (62) then the uniqueness result for homogeneous wave equations discussed in Section 5.1 ensures that

$$H_{A'ABC} = 0, \quad S_{AA'BB'} = 0,$$

in the domain of dependence of  $\mathcal{U}$ . In other words,  $\kappa_{AB}$  solves the Killing spinor equation on the domain of dependence of  $\mathcal{U}$ .  $\square$

**Remark 4.** At first glance one might assume that the standard formulation of the conformal Einstein field equations is the appropriate setting for deriving the conditions obtained in this article. Nevertheless some experimentation reveals that instead of a conformally regular system of wave equations for  $H_{A'ABC}$  and  $S_{AA'BB'}$  one is confronted with an homogeneous Fuchsian system —formally singular at  $\Xi = 0$ . Although one could potentially still analyse this system and obtain an analogous result to Proposition 1, one would require a uniqueness result for solutions to Fuchsian systems of wave equations. Moreover, following the original spirit of the derivation of the conformal Einstein field equations in [9] one is interested in finding conformally regular equations instead of analysing Fuchsian systems. Fortunately, as shown in Section 5 the alternative formulation of the conformal Einstein field equations given in Section 3.2 leads to a regular set of wave equations for  $H_{A'ABC}$  and  $S_{AA'BB'}$ .

## 6 The intrinsic conditions

In this section the conditions (63b)–(63d) are written in terms of intrinsic quantities on  $\mathcal{S}$ . To do so, the space spinor formalism will be exploited. The discussion given in this section is similar to that of [3]. Notice that, nevertheless, in the discussion given in [3] the Einstein field equations are used to simplify expressions associated with the curvature spinors. In the present analysis the curvature spinors are subject to the alternative conformal Einstein field equations as encoded in the zero-quantities (20a)–(20e).



## 6.1 Space spinor formalism

To have a self-contained discussion in this section the space spinor formalism, originally introduced in [29], is briefly recalled —see also [16, 3, 32]. Let  $\tau^{AA'}$  denote the spinorial counterpart of a timelike vector  $\tau^a$ , normal to a spacelike hypersurface  $\mathcal{S}$  and normalised so that  $\tau_a \tau^a = 2$ . Then, it follows that  $\tau_{AA'} \tau^{AA'} = 2$  and, consequently,

$$\tau_{AA'} \tau_B{}^{A'} = \epsilon_{AB}.$$

The covariant derivative  $\nabla_{AA'}$  is then decomposed into the *normal* and *Sen* derivatives:

$$\begin{aligned}\mathcal{P} &\equiv \tau^{AA'} \nabla_{AA'}, \\ \mathcal{D}_{AB} &\equiv \tau_{(A}{}^{A'} \nabla_{B)A'}.\end{aligned}$$

The *Weingarten* spinor and the *acceleration* of the congruence are then defined by

$$\begin{aligned}K_{ABCD} &\equiv \tau_D{}^{C'} \mathcal{D}_{AB} \tau_{CC'}, \\ K_{AB} &\equiv \tau_B{}^{C'} \mathcal{P} \tau_{AC'}.\end{aligned}$$

The above can be inverted to obtain the following formulae which will prove useful in the sequel

$$\begin{aligned}\mathcal{P} \tau_{CC'} &= -K_{CD} \tau^D{}_{C'}, \\ \mathcal{D}_{AB} \tau_{CA'} &= -K_{ABCD} \tau^D{}_{A'}.\end{aligned}$$

The distribution induced by  $\tau_{AA'}$  is integrable if and only  $K^D{}_{(AB)D} = 0$ , in which case  $K_{ABCD}$  describes the extrinsic curvature of the resulting foliation. Nevertheless, this is not required for our subsequent discussion. In other words, we will allow the possibility that the distribution is non-integrable —i.e. the spinor  $K^D{}_{(AB)D}$  will not be assumed to vanish.

Defining the spinors  $\chi_{AB} \equiv K^D{}_{(AB)D}$ ,  $\chi_{ABCD} \equiv K_{(ABCD)}$  and  $\chi \equiv K_{AB}{}^{AB}$ , the Weingarten spinor decomposes as follows

$$K_{ABCD} = \chi_{ABCD} - \frac{1}{2} \epsilon_{A(C} \chi_{D)B} - \frac{1}{2} \epsilon_{B(C} \chi_{D)A} - \frac{1}{3} \chi \epsilon_{A(C} \epsilon_{D)B}. \quad (64)$$

For the following discussion we will also need the commutators form with  $\mathcal{P}$ ,  $\mathcal{D}_{AB}$ . To write these commutators in a succinct way, first define

$$\hat{\square}_{AB} \equiv \tau_A{}^{A'} \tau_B{}^{B'} \square_{A'B'}$$

from which, proceeding analogously as in [3], one obtains

$$[\mathcal{P}, \mathcal{D}_{AB}] = -\frac{1}{2} \chi_{AB} - \square_{AB} + \hat{\square}_{AB} + K_{(A}{}^D \mathcal{D}_{B)D} - K_{AB}{}^{FG} \mathcal{D}_{FG}, \quad (65)$$

$$\begin{aligned}[\mathcal{D}_{AB}, \mathcal{D}_{CD}] &= \frac{1}{2} (\epsilon_{A(C} \square_{D)B} + \epsilon_{B(C} \square_{D)A}) + \frac{1}{2} (\epsilon_{A(C} \hat{\square}_{D)B} + \epsilon_{B(C} \hat{\square}_{D)A}) \\ &\quad + \frac{1}{2} (K_{CDAB} \mathcal{P} - K_{ABCD} \mathcal{P}) + K_{CDF(A} \mathcal{D}_{B)}{}^F - K_{ABF(C} \mathcal{D}_{D)}{}^F\end{aligned} \quad (66)$$

It will also prove convenient to decompose the tracefree Ricci spinor,  $\Phi_{AA'BB'}$  in space spinor form. To do so, introduce its space spinor counterpart  $\Phi_{ABCD} \equiv \tau_B{}^{B'} \tau_D{}^{D'} \Phi_{ACB'D'}$ . The latter can be decomposed as

$$\Phi_{ABCD} = \Theta_{ABCD} + \frac{1}{2} (\epsilon_{C(B} \Phi_{D)A} + \epsilon_{A(B} \Phi_{D)C}) - \frac{1}{3} \Phi \epsilon_{A(B} \epsilon_{D)C} \quad (67)$$

where

$$\Phi \equiv \Phi_A{}^A{}_B{}^B, \quad \Phi_{AB} \equiv \Phi_{(AB)C}{}^C, \quad \Theta_{ABCD} \equiv \Phi_{(ABCD)}$$

## 6.2 Space spinor decompositions and ancillary identities

To obtain intrinsic conditions on  $\mathcal{S}$  from equations (63a)-(63d) we start defining the space spinorial counterpart of the zero quantities  $H_{A'ABC}$ ,  $S_{AA'BB'}$ :

$$H_{ABCD} \equiv \tau_A^{A'} H_{A'BCD}, \quad (68a)$$

$$S_{ABCD} \equiv \tau_A^{B'} \tau_C^{D'} S_{BB'DD'}. \quad (68b)$$

Next, we define the following spinors

$$\begin{aligned} \xi_{AB} &\equiv \mathcal{D}_{(A}^D \kappa_{B)D}, \\ \xi &\equiv \mathcal{D}^{AB} \kappa_{AB}, \\ \xi_{ABCD} &\equiv \mathcal{D}_{(AB} \kappa_{CD)}, \end{aligned}$$

in terms of which we have the following irreducible decomposition

$$H_{CABD} = 3\xi_{ABCD} + \frac{1}{2}(\mathcal{P}\kappa_{BD} + \xi_{BD})\epsilon_{AC} + \frac{1}{2}(\mathcal{P}\kappa_{AD} + \xi_{AD})\epsilon_{BC} - \frac{1}{2}(\mathcal{P}\kappa_{AB} + \xi_{AB})\epsilon_{CD}.$$

Additionally, the following decompositions will prove useful

$$\mathcal{D}_{AB}\kappa_{CD} = \xi_{ABCD} - \frac{1}{2}\epsilon_{A(C}\xi_{D)B} - \frac{1}{2}\epsilon_{B(C}\xi_{D)A} - \frac{1}{3}\xi_{A(C}\epsilon_{D)B}, \quad (69)$$

$$\begin{aligned} \mathcal{D}_{AB}\xi_{CD} &= \frac{1}{6}\epsilon_{AD}\epsilon_{BC}\mathcal{D}_{FG}\xi^{FG} + \frac{1}{6}\epsilon_{AC}\epsilon_{BD}\mathcal{D}_{FG}\xi^{FG} - \frac{1}{4}\epsilon_{BD}\mathcal{D}_{(A}^F\xi_{C)F} \\ &\quad - \frac{1}{4}\epsilon_{BC}\mathcal{D}_{(A}^F\xi_{D)F} - \frac{1}{4}\epsilon_{AD}\mathcal{D}_{(B}^F\xi_{C)F} - \frac{1}{4}\epsilon_{AC}\mathcal{D}_{(B}^F\xi_{D)F} + \mathcal{D}_{(AB}\xi_{CD)}. \end{aligned} \quad (70)$$

Using the definition of  $\xi_{AB}$ , and by commuting derivatives, one obtains the following identities:

$$\mathcal{D}_{AB}\xi^{AB} = -\frac{1}{3}\chi\xi + \frac{1}{2}\chi^{AB}\xi_{AB} - \frac{1}{2}\chi^{AB}\mathcal{P}\kappa_{AB} - \kappa^{AB}\Phi_{AB} + \frac{1}{2}\xi^{ABCD}\chi_{ABCD}, \quad (71a)$$

$$\begin{aligned} \mathcal{D}_{A(B}\xi_{D)A} &= -\frac{2}{3}\mathcal{D}_{BD}\xi + \mathcal{D}_{AC}\xi_{BD}^{AC} + \chi_{(B}^A\mathcal{P}\kappa_{D)A} + \frac{1}{3}\xi\chi_{BD} \\ &\quad + \frac{2}{3}\chi\xi_{BD} - \frac{1}{2}\chi_{(B}^A\xi_{D)A} + \frac{1}{2}\xi^{AC}\chi_{BDAC} - \frac{1}{2}\chi^{AC}\xi_{BDAC} + \xi_{(B}^{ACF}\chi_{D)ACF} \\ &\quad - 4\kappa_{BD}\Lambda - \frac{2}{3}\kappa_{BD}\Phi - \kappa_{(B}^A\Phi_{D)A} + \kappa^{AC}\Theta_{BDAC} + \kappa^{AC}\Psi_{BDAC}, \end{aligned} \quad (71b)$$

$$\begin{aligned} \mathcal{D}_{(AB}\xi_{CD)} &= 2\mathcal{D}_{(A}^F\xi_{BCD)F} + \chi_{(AB}\mathcal{P}\kappa_{CD)} \\ &\quad + \frac{2}{3}\chi\xi_{ABCD} - \frac{1}{3}\xi\chi_{ABCD} + \frac{1}{2}\chi_{(AB}\xi_{CD)} - \chi_{(A}^F\xi_{BCD)F} - \xi_{(A}^F\chi_{BCD)F} \\ &\quad - \xi_{(AB}^{FG}\chi_{CD)FG} - 2\kappa_{(A}^F\Theta_{BCD)F} - \kappa_{(AB}\Phi_{CD)} - 2\kappa_{(A}^F\Psi_{BCD)F}, \end{aligned} \quad (71c)$$

and similarly,

$$\begin{aligned} \mathcal{P}\xi &= -\frac{1}{2}K^{AB}\mathcal{P}\kappa_{AB} + \mathcal{D}^{AB}\mathcal{P}\kappa_{AB} - \frac{1}{3}\chi\xi + K^{AB}\xi_{AB} \\ &\quad + \chi^{AB}\xi_{AB} + 2\kappa^{AB}\Phi_{AB} - \xi^{ABCD}\chi_{ABCD}, \end{aligned} \quad (72a)$$

$$\begin{aligned} \mathcal{P}\xi_{AB} &= 4\kappa_{AB}\Lambda - \frac{2}{3}\kappa_{AB}\Phi - \kappa^{CD}\Psi_{ABCD} - \frac{1}{3}K_{AB}\xi - \frac{1}{3}\xi\chi_{AB} - \frac{1}{3}\chi\xi_{AB} + \frac{1}{2}K^{CD}\xi_{ABCD} \\ &\quad + \frac{1}{2}\chi^{CD}\xi_{ABCD} + \kappa^{CD}\Theta_{ABCD} + \frac{1}{2}\xi^{CD}\chi_{ABCD} + \frac{1}{2}K_{(A}^C\xi_{B)C} - \kappa_{(A}^C\Phi_{B)C} \\ &\quad + \frac{1}{2}\chi_{(A}^C\xi_{B)C} + \xi_{(A}^{CDF}\chi_{B)CDF} + \mathcal{D}_{(A}^C\mathcal{P}\kappa_{B)C} - \frac{1}{2}K_{(A}^C\mathcal{P}\kappa_{B)C}, \end{aligned} \quad (72b)$$

$$\begin{aligned} \mathcal{P}\xi_{ABCD} &= \mathcal{D}_{(AB}\mathcal{P}\kappa_{CD)} - \frac{1}{2}K_{(AB}\mathcal{P}\kappa_{CD)} - \frac{1}{3}\chi\xi_{ABCD} - \frac{1}{3}\xi\chi_{ABCD} - \frac{1}{2}K_{(AB}\xi_{CD)} \\ &\quad + K_{(A}^F\xi_{BCD)F} + 2\kappa_{(A}^F\Psi_{BCD)F} - 2\kappa_{(A}^F\Theta_{BCD)F} - \kappa_{(AB}\Phi_{CD)} \\ &\quad - \frac{1}{2}\chi_{(AB}\xi_{CD)} + \chi_{(A}^F\xi_{BCD)F} - \xi_{(A}^F\chi_{BCD)F} - \xi_{(AB}^{FG}\chi_{CD)FG}. \end{aligned} \quad (72c)$$

**Remark 5.** If the tracefree Ricci spinor  $\Phi_{AA'BB'}$  is made to vanish, then the above identities reduce to those given in [3]. This corresponds to setting  $\Xi = 1$  in the alternative CFEs.

## 6.3 The conditions $H_{A'ABC} = \mathcal{P}H_{A'ABC} = 0$

Given the irreducible decomposition of the zero quantity  $H_{A'ABC}$ , provided above, the condition  $H_{A'ABC} = 0$  is equivalent to

$$\xi_{ABCD} = 0, \quad (73a)$$

$$\mathcal{P}\kappa_{AB} = -\xi_{AB}, \quad (73b)$$

and the condition  $\mathcal{P}H_{A'ABC} = 0$  is equivalent to

$$\mathcal{P}H_{ABCD} = \chi_A{}^F H_{FBCD},$$

which, in turn is equivalent to

$$\mathcal{P}\xi_{ABCD} = 0, \quad (74a)$$

$$\mathcal{P}^2\kappa_{AB} = -\mathcal{P}\xi_{AB}. \quad (74b)$$

The wave equation for  $\kappa_{AB}$ , given in equation (35), can be rewritten in terms of the quantities  $\xi$ ,  $\xi_{AB}$ ,  $\xi_{ABCD}$  as follows

$$\begin{aligned} \mathcal{P}^2\kappa_{BC} = & -2\mathcal{D}_{AD}\xi_{BC}{}^{AD} - \frac{2}{3}\mathcal{D}_{BC}\xi + 2\mathcal{D}_{(B}{}^A\xi_{C)A} - \chi\mathcal{P}\kappa_{BC} - 8\kappa_{BC}\Lambda + 2\kappa^{AD}\Psi_{BCAD} \\ & + \frac{1}{3}K_{BC}\xi + \frac{2}{3}\xi\chi_{BC} + K^{AD}\xi_{BCAD} + 2\chi^{AD}\xi_{BCAD} - K_{(B}{}^A\xi_{C)A} - 2\chi_{(B}{}^A\xi_{C)A}. \end{aligned} \quad (75)$$

It is important to note that the Killing spinor satisfies equation (35) by construction, and therefore (75) can be assumed to hold throughout the domain of dependence of  $\mathcal{U}$ ; in particular, we are free to take further  $\mathcal{P}$ -derivatives of the equation. Through repeated use of the identities (71a)-(71c), (72a)-(72c), along with (73a)-(73b), the above wave equation can be seen to imply (74b), which is therefore trivially satisfied. For future reference note that, using equation (72b), the wave equation for  $\kappa_{AB}$  can alternatively be expressed as

$$\begin{aligned} \mathcal{P}^2\kappa_{AB} + \mathcal{P}\xi_{AB} = & -4\kappa_{AB}\Lambda - \chi\mathcal{P}\kappa_{AB} - \frac{2}{3}\kappa_{AB}\Phi + \kappa^{CD}\Psi_{ABCD} + \frac{1}{3}\xi\chi_{AB} - \frac{1}{3}\chi\xi_{AB} + \frac{3}{2}K^{CD}\xi_{ABCD} \\ & + \frac{5}{2}\chi^{CD}\xi_{ABCD} + \kappa^{CD}\Theta_{ABCD} + \frac{1}{2}\xi^{CD}\chi_{ABCD} - \frac{2}{3}\mathcal{D}_{AB}\xi - 2\mathcal{D}_{CD}\xi_{AB}{}^{CD} \\ & + 2\mathcal{D}_{(A}{}^C\xi_{B)C} - 2\mathcal{D}_{(A}{}^C\mathcal{P}\kappa_{B)C} - \frac{1}{2}K_A{}^C\mathcal{P}\kappa_{B)C} - \kappa_{(A}{}^C\Phi_{B)C} - \frac{3}{2}\chi_{(A}{}^C\xi_{B)C} \\ & + \xi_{(A}{}^{CDF}\chi_{B)CDF}. \end{aligned} \quad (76)$$

Finally, using equations (72c), and (71c) along with (73a)-(73b), one obtains

$$\mathcal{P}\xi_{ABCD} = \kappa_{(A}{}^F\Psi_{BCD)F}.$$

Therefore, equation (74a) is equivalent to imposing the Buchdahl constraint,  $\kappa_{(A}{}^F\Psi_{BCD)F} = 0$ , on  $\mathcal{U}$ .

## 6.4 The conditions $S_{AA'BB'} = \mathcal{P}S_{AA'BB'} = 0$

Using the definition of  $S_{ABCD}$  given in expression (68b) and the expression for  $S_{AA'BB'}$  as given in equation (24), a direct computation using the space spinor formalism introduced in Section 6.1 renders

$$\begin{aligned} S_{ABCD} = & K_{C(D|A|F|}\mathcal{P}\kappa_{B)}{}^F - 6\kappa_{(D}{}^F\Phi_{B)FAC} - \frac{1}{2}K_{ABCD}\xi - \frac{1}{2}K_{CDAB}\xi - K_{CDAF}\xi_B{}^F \\ & - K_{ABCF}\xi_D{}^F + \frac{1}{4}K_C{}^F\mathcal{P}\kappa_{DF}\epsilon_{AB} - \frac{1}{4}\mathcal{P}^2\kappa_{CD}\epsilon_{AB} + \frac{1}{2}\mathcal{P}\xi_{CD}\epsilon_{AB} + \frac{1}{4}K_{CD}\xi\epsilon_{AB} \\ & - \frac{1}{2}K_C{}^F\xi_{DF}\epsilon_{AB} + \frac{1}{4}K_A{}^F\mathcal{P}\kappa_{BF}\epsilon_{CD} - \frac{1}{4}\mathcal{P}^2\kappa_{AB}\epsilon_{CD} + \frac{1}{2}\mathcal{P}\xi_{AB}\epsilon_{CD} + \frac{1}{4}K_{AB}\xi\epsilon_{CD} \\ & - \frac{1}{2}K_A{}^F\xi_{BF}\epsilon_{CD} - \frac{1}{2}\mathcal{P}\xi_{AB}\epsilon_{CD} + \frac{1}{2}\mathcal{D}_{AB}\mathcal{P}\kappa_{CD} + \frac{1}{2}\mathcal{D}_{CD}\mathcal{P}\kappa_{AB} + \frac{1}{2}\epsilon_{CD}\mathcal{D}_{AB}\xi \\ & + \frac{1}{2}\epsilon_{AB}\mathcal{D}_{CD}\xi - \mathcal{D}_{AB}\xi_{CD} - \mathcal{D}_{CD}\xi_{AB}. \end{aligned} \quad (77)$$

Using the decompositions (64), (67) for  $K_{ABCD}$  and  $\Phi_{ABCD}$ , equation (77) can be decomposed in irreducible components. The non-vanishing components (or combinations thereof) of this decomposition are:

$$\begin{aligned} S_{(ABCD)} = & -\xi\chi_{ABCD} - 2\mathcal{D}_{(AB}\xi_{CD)} + \mathcal{D}_{(AB}\mathcal{P}\kappa_{CD)} - 6\kappa_{(A}{}^F\Theta_{BCD)F} \\ & - 3\kappa_{(AB}\Phi_{CD)} - \chi_{(AB}\xi_{CD)} + \frac{1}{2}\chi_{(AB}\mathcal{P}\kappa_{CD)} - 2\xi_{(A}{}^F\chi_{BCD)F} \\ & - \chi_{(ABC}{}^F\mathcal{P}\kappa_{D)F}, \end{aligned} \quad (78a)$$

$$\begin{aligned}
S_{(AB)}^F{}_F - S_{(A}^F|_F|B) = & -\frac{1}{2}\mathcal{P}\kappa_{AB} + \mathcal{P}^2\kappa_{AB} - 2\mathcal{P}\xi_{AB} - 4\kappa_{AB}\Phi - K_{AB}\xi + \frac{2}{3}\chi\xi_{AB} \\
& + 6\kappa^{FC}\Theta_{ABFC} - \mathcal{P}\kappa^{FC}\chi_{ABFC} + 2\xi^{FC}\chi_{ABFC} - 2\mathcal{D}_{AB}\xi \\
& + 2K_{(A}^F\xi_{B)F} - K_{(A}^F\mathcal{D}_{AB}\kappa_{B)F} - 6\kappa_{(A}^F\Phi_{B)F} - 2\chi_{(A}^F\xi_{B)F} \\
& + \chi_{(A}^F\mathcal{P}\kappa_{B)F}, \tag{78b}
\end{aligned}$$

$$\begin{aligned}
S^{FG}{}_{FG} + S^{FG}{}_{GF} = & -2\chi\xi - 2\chi^{FG}\mathcal{P}\kappa_{FG} + 4\chi^{FG}\xi_{FG} - 6\kappa^{FG}\Phi_{FG} + 2\mathcal{D}_{FG}\mathcal{P}\kappa^{FG} \\
& - 4\mathcal{D}_{FG}\xi^{FG}, \tag{78c}
\end{aligned}$$

$$S^{FG}{}_{FG} - S^{FG}{}_{GF} = -K^{FG}\mathcal{P}\kappa_{FG} - 2\mathcal{P}\xi + 2K^{FG}\xi_{FG} + 6\kappa^{FG}\Phi_{FG}. \tag{78d}$$

Note that in deriving expressions (78a)-(78d) the  $H_{ABCD}|_S = 0$  and  $\mathcal{P}H_{ABCD}|_S = 0$  conditions have not been used. Taking into account the  $H_{ABCD}|_S = 0$  conditions, encoded in equations (73a)-(73b) and (74a)-(74b), and exploiting equations (71a)-(71c), the conditions encoded in (78a)-(78d) reduce to

$$\mathcal{P}\xi = \frac{3}{2}K^{FG}\xi_{FG} + 3\kappa^{FG}\Phi_{FG}, \tag{79a}$$

$$\begin{aligned}
\mathcal{P}\xi_{AB} = & \frac{2}{3}\mathcal{D}_{AB}\xi - \frac{4}{3}\kappa_{AB}\Phi - \frac{1}{3}K_{AB}\xi + \frac{1}{3}\chi\xi_{AB} + K_{(A}^F\xi_{B)F} \\
& - \chi_{(A}^F\xi_{B)F} + 2\kappa^{FC}\Theta_{ABFC} - 2\kappa_{(A}^F\Phi_{B)F} + \xi^{FC}\chi_{ABFC}, \tag{79b}
\end{aligned}$$

$$\kappa_{(A}^F\Psi_{BCD)F} = 0. \tag{79c}$$

Furthermore, one can verify, using the identities (72a)-(72b), that equations (79a)-(79b) are identically satisfied if the intrinsic conditions (73a)-(73b) and (74a)-(74b) hold. Additionally, observe that, as discussed in Section 6.3, the vanishing of the Buchdahl constraint (79c) is obtained through condition (74a). In other words, the  $S_{ABCD}|_S = 0$  requirement does not impose any extra conditions than those already encoded in  $H_{ABCD}|_S = 0$  and  $\mathcal{P}H_{ABCD}|_S = 0$ .

Now, to analyse the conditions imposed by requiring  $\mathcal{P}S_{ABCD} = 0$ , observe that

$$\tau_A{}^{B'}\tau_C{}^{D'}\mathcal{P}S_{BB'DD'} = \mathcal{P}S_{ABCD} - K^F{}_C S_{ABFD} - K^F{}_A S_{CDFB}. \tag{80}$$

Consequently, if the conditions  $S_{ABCD}|_S = 0$  hold, then, it is enough to analyse the restriction imposed by  $\mathcal{P}S_{ABCD}|_S = 0$ . Taking a  $\mathcal{P}$ -derivative of equations (78a)-(78d) and exploiting the space spinor formalism one obtains

$$\begin{aligned}
\mathcal{P}S_{(ABCD)} = & -\xi\mathcal{P}\chi_{ABCD} - \chi_{ABCD}\mathcal{P}\xi - 2\mathcal{P}\mathcal{D}_{(AB}\xi_{CD)} + \mathcal{P}\mathcal{D}_{(AB}\mathcal{P}\kappa_{CD)} \\
& - 6\kappa_{(A}^F\mathcal{P}\Theta_{BCD)F} - 3\kappa_{(AB}\mathcal{P}\Phi_{CD)} - \chi_{(AB}\mathcal{P}\xi_{CD)} + \frac{1}{2}\chi_{(AB}\mathcal{P}^2\kappa_{CD)} \\
& - \xi_{(AB}\mathcal{P}\chi_{CD)} - 2\xi_{(A}^F\mathcal{P}\chi_{BCD)F} + 6\Theta_{(ABC}^F\mathcal{P}\kappa_{D)F} - 3\Phi_{(AB}\mathcal{P}\kappa_{CD)} \\
& + 2\chi_{(ABC}^F\mathcal{P}\xi_{D)F} - \chi_{(ABC}^F\mathcal{P}^2\kappa_{D)F} + \frac{1}{2}\mathcal{P}\kappa_{(AB}\mathcal{P}\chi_{CD)} \\
& + \mathcal{P}\kappa_A{}^F\mathcal{P}\chi_{BCD)F}, \tag{81a}
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}(S_{(AB)}^F{}_F - S_{(A}^F|_F|B)) = & \mathcal{P}^3\kappa_{AB} - 2\mathcal{P}^2\xi_{AB} - \frac{1}{3}\mathcal{P}\kappa_{AB}\mathcal{P}\chi - \frac{1}{3}\chi\mathcal{P}^2\kappa_{AB} - 4\kappa_{AB}\mathcal{P}\Phi - K_{AB}\mathcal{P}\xi \\
& + \frac{2}{3}\chi\mathcal{P}\xi_{AB} + 6\kappa^{FC}\mathcal{P}\Theta_{ABFC} - \mathcal{P}\kappa^{FC}\chi_{ABFC} - 2\mathcal{P}\mathcal{D}_{AB}\xi - 4\Phi\mathcal{P}\kappa_{AB} \\
& - \xi\mathcal{P}K_{AB} + \frac{2}{3}\xi_{AB}\mathcal{P}\chi + 2\xi^{FC}\mathcal{P}\chi_{ABFC} + 6\Theta_{ABFC}\mathcal{P}\kappa^{FC} - \chi_{ABFC}\mathcal{P}\xi^{FC} \\
& + 2K_{(A}^F\mathcal{P}\xi_{B)F} - K_{(A}^F\mathcal{P}^2\kappa_{B)F} - 6\kappa_{(A}^F\mathcal{P}\Phi_{B)F} - 2\chi_{(A}^F\mathcal{P}\xi_{B)F} \\
& + \chi_{(A}^F\mathcal{P}^2\kappa_{B)F} - 2\xi_{(A}^F\mathcal{P}K_{B)F} + 2\xi_{(A}^F\mathcal{P}\chi_{B)F} + 6\Phi_{(A}^F\mathcal{P}\kappa_{B)F} \\
& - \mathcal{P}K_{(A}^F\mathcal{P}\kappa_{B)F} - \mathcal{P}\kappa_{(A}^F\mathcal{P}\chi_{B)F}, \tag{81b}
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}(S^{FG}{}_{FG} + S^{FG}{}_{GF}) = & -2\chi\mathcal{P}\xi - 2\mathcal{P}\kappa^{FG}\mathcal{P}\chi_{FG} - 6\kappa^{FG}\mathcal{P}\Phi_{FG} + 2\mathcal{P}\mathcal{D}_{FG}\mathcal{P}\kappa^{FG} - 4\mathcal{P}\mathcal{D}_{FG}\xi^{FG} \\
& - 2\xi\mathcal{P}\chi - 2\chi^{FG}\mathcal{P}^2\kappa_{FG} + 4\chi^{FG}\mathcal{P}\xi_{FG} + 4\xi^{FG}\mathcal{P}\chi_{FG} - 6\Phi^{FG}\mathcal{P}\kappa_{FG}, \tag{81c}
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}(S^{FG}{}_{FG} - S^{FG}{}_{GF}) = & -\mathcal{P}K^{FG}\mathcal{P}\kappa_{FG} - K^{FG}\mathcal{P}^2\kappa_{FG} - 2\mathcal{P}^2\xi + 2K^{FG}\mathcal{P}\xi_{FG} + 6\kappa^{FG}\mathcal{P}\Phi_{FG} \\
& + 2\xi^{FG}\mathcal{P}K_{FG} + 6\Phi^{FG}\mathcal{P}\kappa_{FG}. \tag{81d}
\end{aligned}$$

Observe that, in deriving equations (81a)-(81d), the conditions  $H_{ABCD}|_S = 0$  and  $\mathcal{P}H_{ABCD}|_S = 0$  were not used. Similar to the discussion leading to equations (79a)-(79c), exploiting  $H_{ABCD}|_S = 0$  and  $\mathcal{P}H_{ABCD}|_S = 0$  leads to simpler expressions. To implement this computation it will be convenient to derive some ancillary results first. This is done in the following.

Applying the commutator (65) to  $\mathcal{P}\kappa_{CD}$  and exploiting the intrinsic conditions encoded in (73b) and (74b) one obtains

$$\mathcal{P}\mathcal{D}_{AB}\mathcal{P}\kappa_{CD} + \mathcal{D}_{AB}\mathcal{P}\xi_{CD} = \frac{1}{2}K_{AB}\mathcal{P}\xi_{CD} + \square_{AB}\xi_{CD} - \widehat{\square}_{AB}\xi_{CD} - K_{(A}{}^F\mathcal{D}_{B)F}\xi_{CD} + K_{ABFG}\mathcal{D}^{FG}\xi_{CD}.$$

Similarly, using again the commutator (65) applied now to  $\xi_{CD}$  and exploiting the intrinsic conditions encoded in (73b) and (74b) one obtains

$$\mathcal{P}\mathcal{D}_{AB}\xi_{CD} - \mathcal{D}_{AB}\mathcal{P}\xi_{CD} = -\frac{1}{2}K_{AB}\mathcal{P}\xi_{CD} - \square_{AB}\xi_{CD} + \widehat{\square}_{AB}\xi_{CD} + K_{(A}{}^F\mathcal{D}_{B)F}\xi_{CD} - K_{ABFG}\mathcal{D}^{FG}\xi_{CD}.$$

Comparing the last two expressions one concludes that

$$\mathcal{P}\mathcal{D}_{AB}\mathcal{P}\kappa_{CD} + \mathcal{P}\mathcal{D}_{AB}\xi_{CD} = 0. \quad (82)$$

Additionally, observe that taking a  $\mathcal{P}$ -derivative to equations (72a)-(72b) and exploiting equations (71a)-(71b) along with conditions (73a)-(73b) and (74a)-(74b) we obtain

$$\mathcal{P}^2\xi = \frac{3}{2}K^{AB}\mathcal{P}\xi_{AB} + 3\kappa^{AB}\mathcal{P}\Phi_{AB} + \frac{3}{2}\xi^{AB}\mathcal{P}K_{AB} - 3\xi^{AB}\Phi_{AB}, \quad (83)$$

$$\begin{aligned} \mathcal{P}^2\xi_{AB} = & -\frac{4}{3}\kappa_{AB}\mathcal{P}\Phi - \frac{1}{3}K_{AB}\mathcal{P}\xi + \frac{1}{3}\chi\mathcal{P}\xi_{AB} + K_{(B}{}^C\mathcal{P}\xi_{A)C} + 2\kappa^{CD}\mathcal{P}\Theta_{ABCD} \\ & - 2\kappa_{(B}{}^C\mathcal{P}\Phi_{A)C} - \frac{2}{3}\mathcal{P}\mathcal{D}_{AB}\xi + \frac{1}{3}\xi\mathcal{P}K_{AB} - \chi_{(A}{}^C\mathcal{P}\xi_{B)C} + \frac{1}{3}\xi_{AB}\mathcal{P}\chi + \frac{4}{3}\Phi\xi_{AB} \\ & - \xi_{(A}{}^C\mathcal{P}K_{B)C} + \xi_{(A}{}^C\mathcal{P}\chi_{B)C} + \xi^{CD}\mathcal{P}\chi_{ABCD} - 2\xi^{CD}\Theta_{ABCD} + \xi_A{}^C\Phi_{BC} \\ & + \chi_{ABCD}\mathcal{P}\xi^{CD}. \end{aligned} \quad (84)$$

Using the above expressions along with equations (82), (70), (71a)-(71c), (72a)-(72c), and the  $H_{ABCD}|_S$  and  $\mathcal{P}H_{ABCD}|_S = 0$  conditions encoded in equations (73a)-(73b) and (74a)-(74b) we obtain

$$6\kappa_{(A}{}^F\mathcal{P}\Psi_{BCD)F} + 6\Psi_{(ABC}{}^F\xi_{D)F} = 0, \quad (85a)$$

$$\mathcal{P}^3\kappa_{AB} + \mathcal{P}^2\xi_{AB} = 0. \quad (85b)$$

To simplify equation (85b) we can exploit the wave equation for  $\kappa_{AB}$  as expressed in equation (76). Taking a  $\mathcal{P}$ -derivative of the latter equations and using the identities (71a)-(71c) (72a)-(72c), and the  $H_{ABCD}|_S$  and  $\mathcal{P}H_{ABCD}|_S = 0$  conditions encoded in equations (73a)-(73b) and (74a)-(74b) one obtains

$$\mathcal{P}^3\kappa_{AB} + \mathcal{P}^2\xi_{AB} = 0.$$

Consequently, equation (85a) contains the only independent condition encoded by  $\mathcal{P}S_{AA'BB'}|_S = 0$ . Finally, one can exploit the conformal field equation encoded in the zero-quantity (20e) to express the  $\mathcal{P}$  derivative of the Weyl spinor in terms of intrinsic quantities at  $\mathcal{S}$ . To do so, let

$$\begin{aligned} \Lambda_{ABCD} &\equiv \tau_A{}^{A'}\Lambda_{A'BCD}, \\ Y_{ABCD} &\equiv \tau_D{}^{D'}Y_{ABCD'}. \end{aligned}$$

Exploiting the space spinor formalism one obtains

$$\Lambda_{ABCD} = \frac{1}{2}\mathcal{P}\Psi_{ABCD} - \frac{1}{2}Y_{ABCD} + \mathcal{D}_{QD}\Psi_{ABC}{}^Q,$$

from which one obtains evolution and constraint equations encoded in

$$\Lambda_{(ABCD)} = 0, \quad \Lambda_{AB}{}^Q{}_Q = 0,$$

given explicitly by

$$\mathcal{P}\Psi_{ABCD} + 2\mathcal{D}_{Q(D}\Psi_{ABC)}{}^Q - Y_{ABCD} = 0, \quad (86a)$$

$$\mathcal{D}^{PQ}\Psi_{PQAB} - \frac{1}{2}Y_{AB}{}^Q{}_Q = 0. \quad (86b)$$

Using the evolution equation encoded in expression (86a), the condition given in equation (85a) reads

$$\mathcal{P}S_{(ABCD)} = 6\kappa_{(A}{}^F Y_{BCD)F} + 12\kappa_{(A}{}^F \mathcal{D}_{|F|}{}^G \Psi_{BCD)G} + 6\Psi_{(ABC}{}^F \xi_{D)F} \quad (87)$$

We are now in a position to formulate the main Theorem of this article:

**Theorem 2.** *Consider an initial data set for the (alternative) conformal Einstein field equations, as encoded in the zero-quantities (20a)-(20e), on a spacelike hypersurface  $\mathcal{S}$  and let  $\mathcal{U} \subset \mathcal{S}$  denote an open set. The development of the initial data set will have a Killing spinor in the domain of dependence of  $\mathcal{U}$  if and only if*

$$\mathcal{D}_{(AB}\kappa_{CD)} = 0, \quad (C1)$$

$$\kappa_{(A}{}^F \Psi_{BCD)F} = 0, \quad (C2)$$

$$\kappa_{(A}{}^F Y_{BCD)F} + 2\kappa_{(A}{}^F \mathcal{D}_{|F|}{}^G \Psi_{BCD)G} + \Psi_{(ABC}{}^F \xi_{D)F} = 0, \quad (C3)$$

are satisfied on  $\mathcal{U}$ . The Killing spinor is obtained evolving according to the wave equation (35) with initial data satisfying conditions (C1)-(C3) and

$$\mathcal{P}\kappa_{AB} = -\xi_{AB}. \quad (89)$$

*Proof.* The prior discussion of this section establishes that the conditions

$$\begin{aligned} H_{A'ABC} &= 0, \\ \mathcal{P}H_{A'ABC} &= 0, \\ S_{AA'BB'} &= 0, \\ \mathcal{P}S_{AA'BB'} &= 0 \end{aligned}$$

on  $\mathcal{U} \subset \mathcal{S}$  are equivalent to (C1)-(C3). Hence, appealing to Proposition 1, we see that if (C1)-(C3) hold on  $\mathcal{U}$ , then the domain of dependence of  $\mathcal{U}$  is endowed with a Killing spinor.  $\square$

**Definition.** The equations (C1)-(C3) will be referred to as the *conformal Killing spinor initial data equations*, and a solution,  $\kappa_{AB}$ , thereof a *Killing spinor candidate*.

**Remark 6.** The conditions (C1)-(C3) are a highly-overdetermined system of equations. It therefore follows that, while they are to be read as equations for the Killing spinor candidate,  $\kappa_{AB}$ , the existence of a non-trivial solution to these equations places strong restrictions on the initial data and, consequently, on the resulting spacetime. Observe that (C2) implies that the restriction of the Weyl spinor to  $\mathcal{S}$  is algebraically special. It will be seen in Section 7 that, equation, (C3) places further constraints on curvature associated to initial data for the (alternative) CFEs, in the sense of restricting various components of the Cotton spinor, when expressed in terms of a suitably-adapted spin dyad.

**Remark 7.** While the analysis in this article is carried out via the spinor formalism, we remark here that the main results could alternatively be rewritten in tensorial terms; the above Theorem may be reframed in terms of the existence of a Killing-Yano tensor (rather than of a Killing spinor) on the spacetime development.

The conditions (C1)-(C3) were derived from (63a)-(63d) exploiting the space spinor formalism adapted to a timelike Hermitian spinor  $\tau^{AA'}$  corresponding to the normal vector to the initial hypersurface  $\mathcal{S}$ . Nevertheless, conditions (63a)-(63d) are irrespective of the causal nature of  $\mathcal{S}$ , consequently, a similar analysis to that given in Section 6 can be used to identify spinorial Killing initial data for the conformal Einstein field equations on a timelike or null hypersurface as well.

The initial hypersurface  $\mathcal{S}$  can be chosen to determined by the condition  $\Xi = 0$  so that  $\mathcal{S}$  corresponds to the conformal boundary  $\mathcal{I}$ . In this case, conditions (C1)-(C3) provide with conditions

on *asymptotic initial data* that ensure the existence of a Killing spinor in the development of this data. This Killing spinor can be used to construct a conformal Killing vector in the unphysical spacetime  $(\mathcal{M}, \mathbf{g})$  corresponding to a Killing vector of the physical spacetime  $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$  —see Lemma 1. On the other hand, setting  $\Xi = 1$ , so that we have Cauchy data for the Einstein field equations, the Cotton tensor (spinor) vanishes and conditions (C1)–(C3) reduce to the conditions given in [2, 3, 4]. Note that, while condition (C3) trivialises in this case as it to follow as a consequence of (C1)–(C2) —see [5] for a detailed discussion of this. Nevertheless, in the general case ( $\Xi \neq 1$ ), (C3) encodes non-trivial information about the Cotton spinor and cannot be eliminated by virtue of the conditions (C1)–(C2) alone —see Remark 6.

## 7 Further analysis of the conformal Killing spinor initial data equations

In this section the conditions (C1)–(C3) are further analysed by expressing them in components with respect to a spin dyad adapted to the Killing spinor  $\kappa_{AB}$ . The ultimate goal of this section is to show that, in contrast to the  $\Xi = 1$  case —see [5], the condition (C3) is in general non-trivial; that is to say that it does not follow as a consequence of conditions (C1)–(C2). Rather, we will see that (C3) captures essential information about the Cotton spinor, and may only be eliminated from the conformal Killing spinor initial data equations by additionally constraining certain components of the Cotton spinor.

Recalling that  $Y_{ABCD} = Y_{(ABC)D}$  let  $Y_{\mathbf{i}}$  with  $\mathbf{i} = \mathbf{0}, \dots, \mathbf{7}$  denote the components of  $Y_{ABCD}$  respect to a spin dyad,  $\{o^A, \iota^A\}$ , normalised as  $o_A \iota^A = 1$ . In other words, let

$$\begin{aligned} Y_{\mathbf{0}} &= \iota^A \iota^B \iota^C \iota^D Y_{ABCD}, & Y_{\mathbf{4}} &= \iota^A o^B o^C \iota^D Y_{ABCD}, \\ Y_{\mathbf{1}} &= \iota^A \iota^B \iota^C o^D Y_{ABCD}, & Y_{\mathbf{5}} &= \iota^A o^B o^C o^D Y_{ABCD}, \\ Y_{\mathbf{2}} &= \iota^A \iota^B o^C \iota^D Y_{ABCD}, & Y_{\mathbf{6}} &= o^A o^B o^C \iota^D Y_{ABCD}, \\ Y_{\mathbf{3}} &= \iota^A \iota^B o^C o^D Y_{ABCD}, & Y_{\mathbf{7}} &= o^A o^B o^C o^D Y_{ABCD}. \end{aligned}$$

Using the latter notation  $Y_{ABCD}$  is expressed as follows

$$\begin{aligned} Y_{ABCD} &= Y_{\mathbf{0}} o_A o_B o_C o_D - Y_{\mathbf{1}} o_A o_B o_C \iota_D - 3Y_{\mathbf{2}} o_D o(A o_B \iota_C) + 3Y_{\mathbf{3}} \iota_D o(A o_B \iota_C) \\ &\quad + 3Y_{\mathbf{4}} o_D o(A \iota_B \iota_C) - 3Y_{\mathbf{5}} \iota_D o(A \iota_B \iota_C) - Y_{\mathbf{6}} o_D \iota_A \iota_B \iota_C + Y_{\mathbf{7}} \iota_A \iota_B \iota_C \iota_D. \end{aligned} \quad (90)$$

The results of this section are summarised in the following Proposition:

**Proposition 2.** *If  $\kappa_{AB}\kappa^{AB} \neq 0$  then there exists a dyad,  $\{o, \iota\}$ , and some real-valued function  $\varkappa$  for which*

$$\kappa_{AB} = e^{\varkappa} o_{(A} \iota_{B)}.$$

*In terms of this adapted dyad, and assuming (C1)–(C2), the condition (C3) is then equivalent to*

$$Y_{\mathbf{0}} = Y_{\mathbf{1}} = Y_{\mathbf{6}} = Y_{\mathbf{7}} = 0.$$

*On the other hand, if  $\kappa_{AB}\kappa^{AB} = 0$  then there exists a dyad,  $\{o, \iota\}$ , for which  $\kappa_{AB} = o_A o_B$ , in terms of which condition (C3) is equivalent to*

$$Y_{\mathbf{2}} = Y_{\mathbf{3}} = Y_{\mathbf{4}} = Y_{\mathbf{5}} = Y_{\mathbf{6}} = Y_{\mathbf{7}} = 0.$$

Cases *i*) and *ii*) are dealt with separately in the remainder of this section.

**Remark 8.** Note that if the spacetime is of Type O, i.e.  $\Psi_{ABCD} = 0$ , then it follows from the conformal field equations (namely, the equation  $\Lambda_{A'ABC} = 0$ ) that  $Y_{ABCC'} = 0$  and hence that (C2), (C3) trivialise, leaving only (C1).

## 7.1 Type D Case: $\kappa_{AB}\kappa^{AB} \neq 0$

If  $\kappa_{AB}\kappa^{AB} \neq 0$  then one can choose a normalised spin dyad  $\{o_A, \iota_B\}$  with  $o_A \iota^A = 1$ , adapted to  $\kappa_{AB}$ . In other words, such that

$$\kappa_{AB} = e^\varkappa o_{(A} \iota_{B)}, \quad (91)$$

where  $\varkappa$  is a scalar field. Similarly, condition (C2) implies that

$$\Psi_{ABCD} = \psi o_{(A} o_B \iota_C \iota_{D)}, \quad (92)$$

where  $\psi$  is a scalar field. Using these expressions condition (C1) implies the following equations

$$o^A o^B o^C \mathcal{D}_{BC} o_A = 0, \quad (93a)$$

$$o^A o^B \mathcal{D}_{AB} \varkappa = -2 o^A o^B \iota^C \mathcal{D}_{BC} o_A, \quad (93b)$$

$$o^A \iota^B \mathcal{D}_{AB} \varkappa = \frac{1}{2} o^A o^B \iota^C \mathcal{D}_{AB} \iota_C - \frac{1}{2} o^A \iota^B \iota^C \mathcal{D}_{BC} o_A, \quad (93c)$$

$$\iota^A \iota^B \mathcal{D}_{AB} \varkappa = 2 o^A \iota^B \iota^C \mathcal{D}_{AC} \iota_B, \quad (93d)$$

$$\iota^A \iota^B \iota^C \mathcal{D}_{BC} \iota_A = 0. \quad (93e)$$

Additionally, using equation (91) the spinor  $\xi_{AB}$  can be expressed as

$$\begin{aligned} e^{-\varkappa} \xi_{AB} &= \frac{1}{2} o_{(A} \mathcal{D}_{B)}^C \iota_C - \frac{1}{2} o^C \mathcal{D}_{(A|C|} \iota_{B)} + \frac{1}{2} \iota_{(A} \mathcal{D}_{B)}^C o_C - \frac{1}{2} \iota^C \mathcal{D}_{(A|C|} o_{B)} \\ &\quad - \frac{1}{2} o_{(A} \iota^C \mathcal{D}_{B)C} \varkappa - \frac{1}{2} o^C \iota_{(A} \mathcal{D}_{B)C} \varkappa. \end{aligned} \quad (94)$$

Using equations (92) and (90) the constraint equations encoded in  $\Lambda_{AB}{}^Q{}_Q = 0$  as given by (86b) imply

$$o^A o^B \mathcal{D}_{AB} \psi = 3Y_5 - 3Y_6 - 2o^A \psi \mathcal{D}_{AB} o^B + 4o^A o^B \psi \iota^C \mathcal{D}_{BC} o_A + 2o^A o^B o^C \psi \mathcal{D}_{BC} \iota_A, \quad (95a)$$

$$\begin{aligned} o^A \iota^B \mathcal{D}_{AB} \psi &= -\frac{3}{2} Y_3 + \frac{3}{2} Y_4 - \psi \iota^A \mathcal{D}_{AB} o^B - o^A \psi \mathcal{D}_{AB} \iota^B - \frac{1}{2} o^A o^B \psi \iota^C \mathcal{D}_{AB} \iota_C \\ &\quad - o^A \psi \iota^B \iota^C \mathcal{D}_{AC} o_B + \frac{1}{2} o^A \psi \iota^B \iota^C \mathcal{D}_{BC} o_A + o^A o^B \psi \iota^C \mathcal{D}_{BC} \iota_A, \end{aligned} \quad (95b)$$

$$\iota^A \iota^B \mathcal{D}_{AB} \psi = 3Y_1 - 3Y_2 - 2\psi \iota^A \mathcal{D}_{AB} \iota^B - 4o^A \psi \iota^B \iota^C \mathcal{D}_{AC} \iota_B - 2\psi \iota^A \iota^B \iota^C \mathcal{D}_{BC} o_A, \quad (95c)$$

while condition (C3) is equivalent to

$$\psi o^A o^B o^C \mathcal{D}_{BC} o_A - Y_7 = 0, \quad (96a)$$

$$\psi o^A o^B \iota^C \mathcal{D}_{AB} o_C - \psi o^A \mathcal{D}_{AB} o^B - \frac{1}{6} o^A o^B \mathcal{D}_{AB} \psi + \frac{1}{2} Y_5 - \frac{1}{6} Y_6 = 0, \quad (96b)$$

$$\psi o^A o^B \iota^C \mathcal{D}_{AB} \iota_C - \psi o^A \iota^B \iota^C \mathcal{D}_{BC} o_A - 4\psi \mathcal{D}_{AB} (o^A \iota^B) - 2o^A \iota^B \mathcal{D}_{AB} \psi - 3Y_3 + 3Y_4 = 0, \quad (96c)$$

$$\psi \iota^A \iota^B \iota^C \mathcal{D}_{BC} o_A + \psi \iota^A \mathcal{D}_{AB} \iota^B + \frac{1}{6} \iota^A \iota^B \mathcal{D}_{AB} \psi + \frac{1}{2} Y_2 - \frac{1}{6} Y_1 = 0, \quad (96d)$$

$$\psi \iota^A \iota^B \iota^C \mathcal{D}_{BC} \iota_A - Y_0 = 0. \quad (96e)$$

A computation using equations (94) with (93a)-(93e) and (95a)-(95c), shows that the condition (C3) implies

$$Y_0 = Y_1 = Y_6 = Y_7 = 0. \quad (97)$$

The converse also holds. That is to say, if equation (97) along with (C1)-(C2) are satisfied, and assuming one has initial data for the alternative CFEs —so that, in particular,  $\Lambda_{AB}{}^Q{}_Q = 0$ — then condition (C3) holds.

## 7.2 Type N Case: $\kappa_{AB}\kappa^{AB} = 0$

If  $\kappa_{AB}\kappa^{AB} = 0$  then one can choose a normalised spin dyad  $\{o_A, \iota_B\}$  such that

$$\kappa_{AB} = o_A o_B. \quad (98)$$

Condition (C2) in this adapted dyad implies

$$\Psi_{ABCD} = \psi o_{(A} o_B o_C o_{D)}. \quad (99)$$



Using equation (98) one observes that condition (C1) implies

$$o^A o^B o^C \mathcal{D}_{AB} o_C = 0, \quad (100a)$$

$$o^A o^B \iota^C \mathcal{D}_{(AB} o_{C)} = 0, \quad (100b)$$

$$o^A \iota^B \iota^C \mathcal{D}_{(AB} o_{C)} = 0, \quad (100c)$$

$$\iota^A \iota^B \iota^C \mathcal{D}_{AB} o_C = 0. \quad (100d)$$

Additionally, using equation (98) the spinor  $\xi_{AB}$  can be expressed as

$$\xi_{AB} = -\frac{1}{2} o^C \mathcal{D}_{AC} o_B - \frac{1}{2} o_B \mathcal{D}_{AC} o^C - \frac{1}{2} o^C \mathcal{D}_{BC} o_A - \frac{1}{2} o_A \mathcal{D}_{BC} o^C. \quad (101)$$

Using equations (99) and (90) the constraint equations encoded in  $\Lambda_{AB}{}^Q{}_Q = 0$  as given by equation (86b) imply

$$Y_5 - Y_6 = 0, \quad (102a)$$

$$Y_3 - Y_4 = 0, \quad (102b)$$

$$o^A o^B \mathcal{D}_{AB} \psi = \frac{1}{2} Y_1 - \frac{1}{2} Y_2 - 2o^A \psi \mathcal{D}_{AB} o^B - 2o^A o^B \iota^C \psi \mathcal{D}_{AB} o_C, \quad (102c)$$

Observe that in contrast with the case discussed in Section 7.1, constraints (102a)-(102b) immediately imply algebraic dependence of various components of the Cotton spinor. In this case (C3) is equivalent to

$$Y_5 = Y_7 = 0, \quad (103a)$$

$$o^A o^B o^C \psi \mathcal{D}_{BC} o_A - \frac{1}{2} Y_3 = 0, \quad (103b)$$

$$o^A o^B \psi \iota^C \mathcal{D}_{(AB} o_{C)} - \frac{1}{2} Y_2 = 0. \quad (103c)$$

A computation using equations (101), (100a)-(100d) and (102b)-(102c) shows that condition (C3) implies

$$Y_2 = Y_3 = Y_4 = Y_5 = Y_6 = Y_7 = 0. \quad (104)$$

Again, the converse holds so that condition (C3) may be replaced with equation (104). Collecting together both cases, Proposition 2 follows immediately.

## Conclusions

In this article a *conformal* version of the Killing spinor initial data equations given in [16] are derived. By conformal it is understood that  $(\mathcal{M}, \mathbf{g})$  is conformally related to an Einstein spacetime  $(\hat{\mathcal{M}}, \hat{\mathbf{g}})$ . Consequently, we call these conditions the *conformal Killing spinor initial data equations*. The existence of a non-trivial solution of this system of equations is a necessary and sufficient condition for the existence of a Killing spinor on the development. The conditions are intrinsic to a spacelike hypersurface  $\mathcal{S} \subset \mathcal{M}$ . In the case where the conformal rescaling is trivial,  $\Xi = 1$ , the conditions reduce to those given in [3]. These conditions contain one differential condition and two algebraic conditions. The differential condition corresponds to the so-called *spatial Killing spinor equation*. The first algebraic condition corresponds to the restriction of the Buchdahl constraint on the initial hypersurface and the second imposes restrictions on the Cotton spinor of the initial data set. Moreover, it was shown that, in a spin dyad adapted to the Killing spinor, these conditions can be used along with the conformal Einstein field equations to show that certain components (at least half of them) of the Cotton spinor  $Y_{ABCA'}$  have to vanish on the initial hypersurface  $\mathcal{S}$ . Notice that the conformal approach followed in this article —i.e., use of the (alternative) conformal Einstein field equations— opens the possibility to allow  $\mathcal{S}$  to be determined by  $\Xi = 0$  so that it corresponds to the conformal boundary  $\mathcal{I}$ . The analysis given in this article already shows that in a potential characterisation of the Kerr-de Sitter spacetime, via the existence of Killing spinors at the conformal boundary, the Cotton spinor will play a replant role. This is not unexpected since the conformal boundary of the Kerr-de Sitter spacetime is conformally flat —see [1, 21]. Therefore, the Cotton tensor associated with asymptotic initial data corresponding

to the Kerr-de Sitter spacetime vanishes. Nonetheless, future applications are not restricted to the analysis of de-Sitter like spacetimes. To see this, notice that, the most delicate part of the analysis consisted on finding a system of homogeneous wave equations for  $H_{A'ABC}$  and  $S_{AA'BB'}$ . This system of wave equations in turn, leads to conditions (63a)-(63d) which are irrespective of the causal nature of  $\mathcal{S}$ . Consequently, one could investigate the analogous conditions to those derived in Section 6 considering a timelike or null hypersurface  $\mathcal{S}$  instead. In the latter case one could consider the conformal boundary of an asymptotically flat spacetime. In the case of a timelike hypersurface  $\mathcal{S}$ , the analogous conditions could be useful for the analysis of anti-de Sitter like spacetimes.

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## 8 Appendix

$$\begin{aligned}
P_{C'A'BD} \equiv & -\frac{1}{18}Y_D^{AC}{}_{C'}H_{A'BAC} - \frac{1}{18}Y_B^{AC}{}_{C'}H_{A'DAC} + \frac{1}{36}\Phi_{DAC'B'}S^{AB'}{}_{BA'} \\
& + \frac{1}{36}\Phi_{BAC'B'}S^{AB'}{}_{DA'} - \frac{1}{24}\Phi_{BAA'C'}S^{AB'}{}_{DB'} - \frac{1}{36}\Lambda S_{BA'DC'} + \frac{1}{36}\Phi_{DAC'B'}S_B^{B'A}{}_{A'} \\
& - \frac{1}{24}\Phi_{DAA'C'}S_B^{B'A}{}_{B'} + \frac{1}{36}\bar{\Psi}_{A'C'B'D'}S_B^{B'D}{}_{D'} - \frac{1}{12}\Lambda S_{BC'DA'} - \frac{1}{36}\Lambda S_{DA'BC'} \\
& + \frac{1}{36}\Phi_{BAC'B'}S_D^{B'A}{}_{A'} + \frac{1}{36}\bar{\Psi}_{A'C'B'D'}S_D^{B'B}{}_{D'} - \frac{1}{12}\Lambda S_{DC'BA'} + \frac{1}{36}\Lambda S_B^{B'DB'}\bar{\epsilon}_{A'C'} + \\
& \frac{1}{36}\Lambda S_D^{B'B}{}_{BB'}\bar{\epsilon}_{A'C'} + \frac{1}{9}H_{B'BDC}\nabla_{AC'}\Phi^{AC}{}_{A'B'} - \frac{1}{9}\Psi_{BDCF}\nabla_{AC'}H_{A'}^{ACF} \\
& + \frac{1}{3}\Lambda\nabla_{AC'}H_{A'BD}{}^A - \frac{1}{3}H_{A'BDA}\nabla^A{}_{C'}\Lambda + \frac{1}{9}\Phi_D^A{}_{A'}\nabla_{CC'}H_{B'BA}{}^C \\
& + \frac{1}{9}\Phi^{AC}{}_{A'}\nabla_{CC'}H_{B'BDA} + \frac{1}{9}\Phi_B^A{}_{A'}\nabla_{CC'}H_{B'DA}{}^C - \frac{1}{9}H_{B'DAC}\nabla^C{}_{C'}\Phi_B^A{}_{A'} \\
& - \frac{1}{9}H_{B'BAC}\nabla^C{}_{C'}\Phi_D^A{}_{A'} - \frac{1}{9}H_{A'}^{ACF}\nabla_{FC'}\Psi_{BDAC} + \frac{1}{9}\Psi_{DACF}\nabla^F{}_{C'}H_{A'B}{}^{AC} \\
& + \frac{1}{9}\Psi_{BACF}\nabla^F{}_{C'}H_{A'D}{}^{AC}
\end{aligned}$$

$$\begin{aligned}
Q_{C'A'BD} \equiv & \frac{1}{18}\Phi_{DAC'B'}S^{AB'}{}_{BA'} + \frac{1}{18}\Phi_{BAC'B'}S^{AB'}{}_{DA'} - \frac{1}{12}\Phi_{BAA'C'}S^{AB'}{}_{DB'} - \frac{1}{18}\Lambda S_{BA'DC'} \\
& + \frac{1}{18}\Phi_{DAC'B'}S_B^{B'A}{}_{A'} - \frac{1}{12}\Phi_{DAA'C'}S_B^{B'A}{}_{B'} + \frac{1}{18}\bar{\Psi}_{A'C'B'D'}S_B^{B'D}{}_{D'} - \frac{1}{6}\Lambda S_{BC'DA'} \\
& - \frac{1}{18}\Lambda S_{DA'BC'} + \frac{1}{18}\Phi_{BAC'B'}S_D^{B'A}{}_{A'} + \frac{1}{18}\bar{\Psi}_{A'C'B'D'}S_D^{B'B}{}_{D'} - \frac{1}{6}\Lambda S_{DC'BA'} \\
& + \frac{1}{12}\Psi_{BDAC}S^{AB'C}{}_{B'}\bar{\epsilon}_{A'C'} - \frac{7}{36}\Lambda S_B^{B'DB'}\bar{\epsilon}_{A'C'} - \frac{1}{36}\Lambda S_D^{B'B}{}_{BB'}\bar{\epsilon}_{A'C'} \\
& - \frac{1}{12}\Lambda S^{AB'}{}_{AB'}\epsilon_{BD}\bar{\epsilon}_{A'C'} - \frac{1}{18}H_{B'DAC}\nabla_{BC'}\Phi^{AC}{}_{A'B'} + \frac{1}{18}H_{A'}^{ACF}\nabla_{BC'}\Psi_{DACF} \\
& + \frac{1}{18}\Psi_{DACF}\nabla_{BC'}H_{A'}^{ACF} - \frac{1}{18}\Phi^{AC}{}_{A'}\nabla_{BC'}H_{B'DAC} - \frac{1}{18}H_{B'BAC}\nabla_{DC'}\Phi^{AC}{}_{A'} \\
& + \frac{1}{18}H_{A'}^{ACF}\nabla_{DC'}\Psi_{BACF} + \frac{1}{18}\Psi_{BACF}\nabla_{DC'}H_{A'}^{ACF} - \frac{1}{18}\Phi^{AC}{}_{A'}\nabla_{DC'}H_{B'BA}{}^C
\end{aligned}$$

$$\begin{aligned}
U_{A'BC'D} \equiv & -\frac{1}{24}Y_D^{AC}{}_{C'}H_{A'BAC} - \frac{1}{24}Y_B^{AC}{}_{C'}H_{A'DAC} - \frac{1}{24}Y_D^{AC}{}_{A'}H_{C'BAC} - \frac{1}{24}Y_B^{AC}{}_{A'}H_{C'DAC} \\
& - \frac{1}{12}\Psi_{BDCF}\nabla_{AC'}H_{A'}^{ACF} - \frac{1}{12}H_{C'}^{ACF}\nabla_{FA'}\Psi_{BDAC} - \frac{1}{12}H_{A'}^{ACF}\nabla_{FC'}\Psi_{BDAC} \\
& + \frac{1}{12}\Psi_{DACF}\nabla^F{}_{C'}H_{A'B}{}^{AC} + \frac{1}{12}\Psi_{BACF}\nabla^F{}_{C'}H_{A'D}{}^{AC} - \frac{1}{4}\Psi_{BDAC}S^{(A}{}_{(A'}{}^{C)}{}_{C')}
\end{aligned}$$

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